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Quadratic Hedging
in Continuous Incomplete Markets

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Abstract

In financial mathematics the Black-Scholes model is well known for pricing a European call or put option. This model was the foundation to study the pricing and hedging of general contingent claims in complete markets. As a result it has been shown that any contingent claim can be exactly replicated by a dynamic, self-financing trading strategy, which trades only in the underlying assets. Unfortunately, in reality there are simply not enough assets to deal with every cause of uncertainty in the market. Hence, one is confronted with incomplete markets and if one tries to hedge a contingent claim an intrinsic risk remains.

In this thesis we will look at a quadratic approach to study and compare the riskiness of hedging strategies for a discounted contingent claim H. We use a quadratic approach, since we do not know in advance if we deal with a buyer or seller. Furthermore, with this approach we are able to work with the L^2-space, which has nice mathematical properties.
Even if the theory was already developed in the late 90's, quadratic hedging finds its use in most recent topics such as in term structure modelling with stochastic discontinuities, see [Fontana et al., 2024] and [Fontana et al., 2020].

As a first approach we will rely on the constraint that the terminal portfolio value matches the contingent claim H and we try to minimize the conditional mean squared error of the remaining costs of the strategy. We will elaborate this idea in the case where the underlying discounted, d-dimensional, real valued price process X, which models the price vector of d discounted stocks, is a local martingale and later in the general case, where X is a semimartingale. This quadratic hedging approach is called **local risk-minimization**. We will see, that in the martingale case the **Galtchouk-Kunita-Watanabe decomposition** of the claim H will play an important role and later, in the general case, the so called **Föllmer-Schweizer decomposition**. Furthermore, we assume that there exists at least one equivalent local martingale measure, under which the semimartingale X is a local martingale. This assumption can be interpreted as an absence of arbitrage. We will see that the **minimal equivalent local martingale measure** $\mathbb{Q}_T$ will play an important role in finding the Föllmer-Schweizer decomposition of H.
As a second idea, we will rely on self-financing strategies and try to minimize the squared mean between the terminal portfolio value of our strategy and the contingent claim H. This quadratic hedging idea is called **mean-variance hedging** and it is heavily connected to the topic of closedness of spaces of stochastic integrals.

This thesis emphasizes on the first approach, since, intuitively, in practice one prefers to be able to adjust the risk until the maturity of the contingent claim H.

The first chapter briefly recaps the fundamentals of market models in continuous time and points out the difference between a complete and an incomplete market. Chapter two works out the above introduced two hedging approaches in the martingale and the more complexe semimartingale case. Finally, in the third chapter an application to a portfolio of unit-linked life insurance contracts can be found and in chapter four we concluded the key ideas and results of this thesis.

Kurzfassung

In der Finanzmathematik ist das Black-Scholes-Modell für die Bewertung einer europäischen Kauf- oder Verkaufsoption sehr bekannt. Dieses Modell war die Grundlage für die Untersuchung der Preisbildung und Absicherung allgemeiner Eventualforderungen auf vollständigen Märkten. So konnte gezeigt werden, dass jede Eventualforderung durch eine dynamische, sich selbst finanzierende Handelsstrategie, die nur mit den zugrunde liegenden Vermögenswerten handelt, exakt nachgebildet werden kann. Leider gibt es in der Realität einfach nicht genügend Vermögenswerte, um alle Unsicherheitsfaktoren auf dem Markt zu berücksichtigen. Daher ist man mit unvollständigen Märkten konfrontiert, und wenn man versucht, eine Eventualforderung abzusichern, was auch als hedging bezeichnet wird, bleibt ein intrinsisches Risiko bestehen.

In dieser Arbeit wird ein quadratischer Ansatz zur Untersuchung und zum Vergleich des Risikos von Absicherungsstrategien für eine diskontierte Eventualforderung H betrachtet. Wir verwenden einen quadratischen Ansatz, da wir im Voraus nicht wissen, ob wir es mit einem Käufer oder Verkäufer zu tun haben. Außerdem können wir mit diesem Ansatz mit dem L^2-Raum arbeiten, der schöne mathematische Eigenschaften hat.
Auch wenn die Theorie bereits in den späten 90er Jahren entwickelt wurde, findet das quadratische Hedging seine Anwendung in neueren Themen wie z.B. in der Zins-Struktur-Modellierung mit stochastischen Unstetigkeiten, siehe [Fontana et al., 2024] und [Fontana et al., 2020].

Als ersten Ansatz werden wir uns auf die Bedingung stützen, dass der Endwert des Portfolios mit der Eventualforderung H übereinstimmt, und wir versuchen, den bedingten mittleren quadratischen Fehler der verbleibenden Kosten der Strategie zu minimieren. Wir werden diese Idee für den Fall ausarbeiten, dass der zugrunde liegende diskontierte, d-dimensionale, reelle Preisprozess X, der den Preisvektor von d diskontierten Aktien modelliert, ein lokales Martingal ist, und später für den allgemeinen Fall, dass X ein Semimartingal ist. Dieser quadratische Absicherungsansatz wird als **lokale Risikominimierung** bezeichnet. Wir werden sehen, dass im Martingal-Fall die **Galtchouk-Kunita-Watanabe-Zerlegung** der Forderung H eine wichtige Rolle spielt und später, im allgemeinen Fall, die sogenannte **Föllmer-Schweizer-Zerlegung**. Weiters nehmen wir an, dass es mindestens ein äquivalentes lokales Martingalmaß gibt, unter dem das Semimartingal X ein lokales Martingal ist. Diese Annahme kann als Abwesenheit von Arbitrage interpretiert werden. Wir werden sehen, dass das **minimale äquivalente lokale Martingalmaß** $\mathbb{Q}_T$ eine wichtige Rolle bei

der Suche nach der Föllmer-Schweizer-Zerlegung von H spielen wird.
Als zweite Idee werden wir uns auf selbstfinanzierende Strategien stützen und versuchen, den quadratischen Erwartungswert zwischen dem Endportfoliowert unserer Strategie und der Eventualforderung H zu minimieren. Diese quadratische Hedging-Idee wird als **Mean-Variance Hedging** bezeichnet und ist eng mit dem Thema der Abgeschlossenheit von Räumen stochastischer Integrale verbunden.

In dieser Arbeit wird der Schwerpunkt auf den ersten Ansatz gelegt, da man in der Praxis intuitiv die Möglichkeit bevorzugt, das Risiko bis zur Fälligkeit der Eventualforderung H anpassen zu können.

Das erste Kapitel fasst kurz die Grundlagen von Marktmodellen in kontinuierlicher Zeit zusammen und zeigt den Unterschied zwischen einem vollständigen und einem unvollständigen Markt auf. Im zweiten Kapitel werden die beiden oben vorgestellten Absicherungsansätze für den Martingal- und den komplexeren Semimartingal-Fall ausgearbeitet. Schließlich findet sich im dritten Kapitel eine Anwendung auf ein Portfolio fondsgebundener Lebensversicherungsverträge, und in Kapitel vier werden die wichtigsten Ideen und Ergebnisse dieser Arbeit zusammengefasst.

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Contents

1 Market models in continuous time

The following chapter is structurally based on [Müller, 2022]. Details about mathematical finance and mathematics of arbitrage can be found in [Karatzas and Shreve, 1998] and [Delbaen and Schachermayer, 2006].
Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space equipped with a filtration $\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ on a finite time horizon $T > 0$. Assume that $\mathcal{F}$ satisfies the usual conditions, namely right continuity und $\mathbb{P}$-completeness. This means, that $\mathcal{N} := \{A \in \mathcal{A} \,|\, \mathbb{P}(A) = 0\} \subseteq \mathcal{F}_0$.
Let the market consist of $d+1$ tradeable assets with real-valued price processes $\tilde{S}^i = (\tilde{S}^i_t)_{0 \leq t \leq T}$, for $i = 0, \ldots, d$. $\tilde{S}^0 > 0$ represents the riskless bond or bank account. Set $\tilde{S} = (\tilde{S}^0, \ldots, \tilde{S}^d)$. We use $\tilde{S}^0$ as a numèraire and call $S = (S)_{0 \leq t \leq T}$ with

$$S_t = (1, S^1_t, \ldots, S^d_t), \quad S^i_t = \tilde{S}^i_t / \tilde{S}^0_t$$

the discounted price process.
To apply basic concepts of stochastic analysis, we further assume that each $\tilde{S}^i$ is a *semimartingale*. Each process $\tilde{S}^i$ is assumed to be adapted to the filtration $\mathcal{F}$ with càdlàg paths and the stochastic integral $\int Y \, d\tilde{S}^i$ exists and is well defined for a previsible and $\tilde{S}^i$-integrable process $Y \in L^1(\tilde{S}^i)$. For more details we refer to [Protter, 2005].

Example. One well known continuous market model is the **Black-Scholes Model**. The riskless bond is described by $\tilde{S}^0_t = e^{rt}$ and the stock price by

$$\tilde{S}^1_t = \tilde{S}^1_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B_t\right) \quad (0 \leq t \leq T),$$

where $\tilde{S}^1_0 > 0$ denotes the initial stock value and $B = (B_t)_{0 \leq t \leq T}$ a Brownian motion. The parameters fulfill $r \geq 0$, $\mu \in \mathbb{R}$, and $\sigma > 0$. Here, r refers to the interest rate, μ to the drift and σ to the volatility. As probability space we use the space on which B is defined and set $\mathcal{F} = \sigma(\mathcal{F}^B \cup \mathcal{N})$. This $\mathbb{P}$-completed canonical filtration of B satisfies the usual conditions.

1.1 Self-financing strategies

Definition 1.1 (trading strategy, value process, cumulative gains, self-financing). *A **trading strategy** $\phi = (\phi_t)_{0 \leq t \leq T}$ is a $(d+1)$-dimensional real-valued process,* *where ϕ^i is previsible for $i = 0, \ldots, d$. Its* ***value process*** $\tilde{V}(\phi) = (\tilde{V}_t(\phi))_{0 \leq t \leq T}$ *is defined by*

$$\tilde{V}_t(\phi) := \phi_t \cdot \tilde{S}_t = \sum_{i=0}^{d} \phi_t^i \tilde{S}_t^i.$$

The ***cumulative gains*** *of ϕ up to time t are*[1]

$$\int_0^t \phi_s \cdot d\tilde{S}_s := \sum_{i=0}^{d} \int_0^t \phi_s^i \, d\tilde{S}_s^i.$$

A trading strategy ϕ is called ***self-financing*** *if*

$$\int_0^t \phi_s \cdot d\tilde{S}_s = \tilde{V}_t(\phi) - \tilde{V}_0(\phi) \quad (0 \leq t \leq T). \tag{1.1.1}$$

Remark. By the above defintion a trading strategy is self-financing if

$$d\tilde{V}_t(\phi) = \phi_t \cdot d\tilde{S}_t := \sum_{i=0}^{d} \phi_t^i \, d\tilde{S}_t^i \quad (0 \leq t \leq T).$$

The intuitive meaning of a self-financing strategy is that it is self-supporting: after time zero no further capital is added or removed from the portfolio and the initial portfolio value $\tilde{V}_0(\phi) = \phi_0 \cdot \tilde{S}_0$ is continuously rebalanced up to time t. Thus, changes of $\tilde{V}(\phi)$ are only due to changes of $\tilde{S}$ and we have

$$\tilde{V}_t(\phi) = \tilde{V}_0(\phi) + \int_0^t \phi_s \cdot d\tilde{S}_s \quad (0 \leq t \leq T).$$

Lemma 1.1. *ϕ is self-financing if and only if $dV_t(\phi) = \phi_t \cdot dS_t \; \forall \, 0 < t \leq T$.*

Proof. The assertion holds true for a general numeraire $N := \tilde{S}^0$, which is a strictly positive semimartingale. Using Itô's Formula and the product rule we get that $V_t(\phi)$, N^{-1} and all S_t^i are semimartingales. Further, the theorem for jumps of a stochastic integral gives

$$\Delta\tilde{V}_t(\phi) = \Delta\left(\int_0^t \phi_s \cdot d\tilde{S}_s\right) = \phi_t \cdot \Delta\tilde{S}_t \quad (0 < t \leq T),$$

where $\Delta\tilde{V}_t(\phi) := \tilde{V}_t(\phi) - \tilde{V}_{t-}(\phi)$. By comparison of coefficients we get $\tilde{V}_{t-}(\phi) = \phi_t \cdot \tilde{S}_{t-}$. Using the properties of the quadratic variation we get

$$\langle \tilde{V}(\phi), N^{-1} \rangle = \left\langle \int_0^{\cdot} \phi \cdot d\tilde{S}, N^{-1} \right\rangle = \int_0^{\cdot} \phi \cdot d\langle \tilde{S}, N^{-1} \rangle.$$

[1] All stochastic integrals are Itô integrals.

Finally the product rule yields for $0 < t \le T$

$$\begin{aligned} dV_t(\phi) &= d(N^{-1}\tilde{V}_t(\phi)) \\ &= N_-^{-1}\, d\tilde{V}_t(\phi) + \tilde{V}_{t-}(\phi)\, dN^{-1} + d\langle \tilde{V}(\phi), N^{-1}\rangle_t \\ &= N_-^{-1}\phi_t \cdot d\tilde{S}_t + \phi_t \cdot \tilde{S}_{t-}\, dN^{-1} + \phi_t \cdot d\langle \tilde{S}, N^{-1}\rangle_t \\ &= \phi_t \cdot \left(N_-^{-1}\, d\tilde{S}_t + \tilde{S}_{t-}\, dN^{-1} + d\langle \tilde{S}, N^{-1}\rangle_t \right) \\ &= \phi_t \cdot d(N^{-1}\tilde{S}_t) \\ &= \phi_t \cdot dS_t, \end{aligned}$$

if and only if ϕ is self-financing. □

Remark. $dV_t(\phi) = \phi_t \cdot dS_t \;\forall\, 0 < t \le T$ is equivalent to

$$\phi_t^0 + \sum_{i=1}^d \phi_t^i S_t^i - V_0(\phi) = V_t(\phi) - V_0(\phi) = \int_0^t \phi_s \cdot dS_s = \sum_{i=1}^d \int_0^t \phi_s^i\, dS_s^i. \tag{1.1.2}$$

Since $dS^0 = 0$, this equation implies that ϕ^0 is uniquely determined by $V_0(\phi)$ and $(\phi^1, \dots, \phi^d)$.

1.2 Arbitrage

Definition 1.2 **(admissible).** *We call a trading strategy ϕ* ***admissible****, if $V(\phi)$ is bounded from below by a constant.*

Remark. This means $V(\phi) \ge -K$, for K constant. Intuitively, this condition ensures that an investor can only use the trading stragy ϕ if he has at least a credit amount of K. Unfortunately, short selling of one unit of asset i is not admissible, since $V_t(\phi) = -e_i \cdot S_t = -S_t^i$ is unbounded from below. We will later transfer to L^2-*admissible* trading strategies, where short selling is included.

Definition 1.3 **(equivalent martingale measure).** *A probability measure $\mathbb{Q}$ is called* ***equivalent martingale measure*** *if $\mathbb{Q}$ and $\mathbb{P}$ have the same zero sets, which is denoted by $\mathbb{Q} \sim \mathbb{P}$, and S is a local martingale with respect to $\mathbb{Q}$.*

Remark. If ϕ is admissible and self-financing we have that $V(\phi)$ is a local martingale with respect to $\mathbb{Q}$. Since $V(\phi)$ is bounded from below we even have that it is a supermartingale.

Definition 1.4 **(arbitrage).** *An admissible, self-financing trading strategy ϕ with*

$$\tilde{V}_0(\phi) \le 0, \quad \tilde{V}_T(\phi) \ge 0, \quad \mathbb{P}(\tilde{V}_T(\phi) > 0) > 0$$

is called an ***arbitrage****.*

Theorem 1.2 (**First Fundamental Theorem of Asset Pricing**). *If there exists an equivalent local martingale measure $\mathbb{Q}$ for S, then the market model is arbitrage free.*

Proof. Assume that ϕ is an admissible, self-financing trading strategy with $\tilde{V}_0(\phi) \leq 0$ and $\tilde{V}_T(\phi) \geq 0$. S is by assumption a local $\mathbb{Q}$-martingale. By the previous remark we have that $V(\phi)$ is a $\mathbb{Q}$-supermartingale. This implies $\mathbb{E}_{\mathbb{Q}}(V_T(\phi)) \leq \mathbb{E}_{\mathbb{Q}}(V_0(\phi)) \leq 0$. Together with the assumption $\tilde{V}_T(\phi) \geq 0$ it follows that $\tilde{V}_T(\phi) = 0$ $\mathbb{Q}$-almost surely. Since $\mathbb{Q} \sim \mathbb{P}$, we also have that $\tilde{V}_T(\phi) = 0$ $\mathbb{P}$-almost surely. Hence, there exists no arbitrage. □

Remark. The reverse is also true, but in most cases only the given direction is practically useful.

1.3 Pricing and hedging

Two major aspects of financial mathematics are pricing and hedging of **contingent claims**. We restrict ourselfs to claims, which pay only at maturity T. Its payoff is described by an $\mathcal{F}_T$-measurable random variable $\tilde{H} \geq 0$.

Remark. Note that the payoff $\tilde{H}$ may depend on the history of $\tilde{S}$ up to time T.

Typical examples are European call and put options on the asset i with strike price K and maturity T. The net payoff is the random amount

$$\tilde{H}^{call} := \left(\tilde{S}^i_T - K\right)_+, \quad \tilde{H}^{put} := \left(K - \tilde{S}^i_T\right)_+ .$$

An example of a payoff depending on the history of $\tilde{S}^i$ would be

$$\tilde{H} := \left(\tilde{S}^i_T - \frac{1}{T}\int_0^T \tilde{S}^i_u \, du\right)_+ ,$$

which is a call option on the average stock value of asset i.

Definition 1.5 (**attainability, replicating strategy**). *A contingent claim $\tilde{H}$ is called **attainable** if there is an admissible, self-financing trading strategy ϕ such that $\tilde{V}_T(\phi) = \tilde{H}$ $\mathbb{P}$-a.s.. In this case ϕ is called **replicating strategy** of $\tilde{H}$.*

Suppose $\tilde{H}$ is an attainable claim and ϕ is its replicating strategy. Let $\mathbb{Q}$ be an equivalent martingale measure. In this case is $V(\phi)$ a supermartingale with respect to $\mathbb{Q}$ and

$$\mathbb{E}_{\mathbb{Q}}(H \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}(V_T(\phi) \,|\, \mathcal{F}_t) \leq \operatorname*{ess\,inf}_{\psi} V_t(\psi),$$

where $H := \tilde{H}/\tilde{S}^0$.

Definition 1.6 (**L^2-admissible**). *A trading strategy ϕ is called L^2-**admissible** with respect to an equivalent martingale measure $\mathbb{Q}$ if*

$$\mathbb{E}_{\mathbb{Q}}\left(\sum_{i,j=0}^{d}\int_0^T \phi_s^i\phi_s^j\, d\langle S^i, S^j\rangle_s\right) < \infty.$$

By Itô's isometry is $V(\phi)$ not only a $\mathbb{Q}$-supermartingale, but also an L^2-bounded $\mathbb{Q}$-martingale, if ϕ is an L^2-admissible trading strategy. In this case we have

$$V_t(\phi) = \mathbb{E}_{\mathbb{Q}}(V_T(\phi)\,|\,\mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}(H\,|\,\mathcal{F}_t).$$

Remark. Short selling is included for L^2-admissible trading strategies, since it is vector space. Thus, $-\phi$ is L^2-admissible if we assume ϕ is L^2-admissible.

Definition 1.7 (**fair price, hedge, complete market**). *The **faire price** of an attainable claim $\tilde{H}$ at time t is given by*

$$\operatorname*{ess\,inf}_{\phi} V_t(\phi),$$

*where ϕ runs over all replicating strategies of $\tilde{H}$. ϕ is called a **hedge** of $\tilde{H}$, if ϕ is a replicating strategy with $V(\phi)$ being a $\mathbb{Q}$-martingale. In this case the discounted fair price is given by*

$$\mathbb{E}_{\mathbb{Q}}(H\,|\,\mathcal{F}_t).$$

*A continuous market model is called **complete**, if every bounded claim has a hedge.*

Theorem 1.3 (**Second Fundamental Theorem of Asset Pricing**). *Assume that $\mathcal{F}_0$ is $\mathbb{P}$-trivial. Then an arbitrage free time continuous market model is complete if and only if the equivalent martingale measure is unique on $\mathcal{F}_T$.*

Proof. We will proof only the easy direction. The other direction can be found in [Jarrow, 2018].
Assume the market model is complete and let $H = I_A$ with $A \in \mathcal{F}_T$. Then there exists a hedge ϕ with $V_T(\phi) = I_A$, such that $V_0(\phi) = \mathbb{E}_{\mathbb{Q}}(V_T(\phi)\,|\,\mathcal{F}_0) = \mathbb{E}_{\mathbb{Q}}(I_A\,|\,\mathcal{F}_0)$. Since $\mathcal{F}_0$ is $\mathbb{P}$-trivial, the equation is $\mathbb{P}$-a.s. (and also $\mathbb{Q}$-a.s.) constant and we get

$$\mathbb{Q}(A) = \mathbb{E}_{\mathbb{Q}}(I_A\,|\,\mathcal{F}_0) = V_0(\phi) = \mathbb{E}(V_0(\phi)).$$

Let ϕ' be another hedge. Then $V_0(\phi)$ and $V_0(\phi')$ differ only on a zero set and $\mathbb{E}(V_0(\phi)) = \mathbb{E}(V_0(\phi'))$, thus $\mathbb{Q}(A)$ is unique. □

A more general setting, where we do not rely on $\mathcal{F}_0$ beeing $\mathbb{P}$-trivial can be found in [Harrison and Pliska, 1983] or [Delbaen and Schachermayer, 2006].

Incomplete markets

Assume we have an arbitrage free, but incomplete market. In such a market there exist multiple equivalent martingale measures. The set of all equivalent martingale measures is a convex set and denoted by $\mathcal{Q}$. For each $\mathbb{Q}$ we have a system of arbitrage free prices. If we add a $\mathbb{Q}$-integrable claim H as additional asset to the model, whose price lies within the interval

$$\left[\operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}(H \mid \mathcal{F}_t),\ \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}(H \mid \mathcal{F}_t)\right],$$

then the enlarged model is again arbitrage free. Every other choice outside the interval leads to an arbitrage opportunity. One can prove, that at the left end point the buyer has no risk and at the right end point the seller has no risk. Thus, buyer and seller have to agree on a price inbetween.
One idea is to introduce certain criteria to determine the price and the replicating strategy of a claim in an incomplete market. We will focus on two quadratic criteria in the next chapter, namely **local risk-minimization** and **mean-variance hedging**. In practice one often chooses the first approach, since one prefers to bear the risk *before* the maturity of the claim and not *at* maturity, as in the second approach. Therefore, we will emphasis our work on local risk-minimization and only rigorously introduce the concepts of mean-variance hedging.

2 Quadratic hedging

Assume, that we have an incomplete, arbitrage free, time continuous market. For simplicity we define

$$S = (1, S^1, \dots, S^d) =: (1, X),$$
$$\phi = (\phi^0, \phi^1, \dots, \phi^d) =: (\phi^0, \xi).$$

Since perfect hedging in the sense of Definition 1.7 is not possible in an incomplete market, we have to be able to make adjustments at each time $t \in [0, T]$ to compensate the ocurring hedging error. Therefore, we weaken the condition of previsibility of ϕ^0 to adaptedness and since $dS^0 = 0$, no problems arise in the definition of the stochastic integral. Furthermore, we are only considering L^2-admissible trading strategies. All subsequent appearing processes are assumed to be real-valued.

2.1 The martingale case

The following section is based on [Föllmer and Sondermann, 1986, Schweizer, 1999]. We start by discussing the two quadratic approaches in the simplified case, where the adapted, càdlàg, d-dimensional process X is already a square integrable local martingale with respect to $\mathbb{P}$[1]. Further, note that $\mathcal{F}_0$ may *not* be $\mathbb{P}$-trivial. Hence, instead of a deterministic starting value we have a random variable.

Definition 2.1 (strategy). *We now call $\phi = (\phi^0, \xi)$ with $\phi^0 = (\phi^0_t)_{0 \le t \le T}$ and $\xi = (\xi_t)_{0 \le t \le T}$ a **strategy**, if ϕ^0 is adapted to the filtration $\mathcal{F}$ and ξ is a d-dimensional previsible process satisfying*

$$\mathbb{E}\left(\int_0^T \xi_s^{tr}\, d\langle X \rangle_s\, \xi_s \right) < \infty,$$

where[2]

$$\int_0^T \xi_s^{tr}\, d\langle X \rangle_s\, \xi_s := \sum_{i,j=1}^d \int_0^T \xi_s^i \xi_s^j\, d\langle X^i, X^j \rangle_s,$$

*such that the discounted **value process** $V_t = \phi^0_t + \xi \cdot X_t$ is càdlàg and square-integrable.*

[1] This means $\mathbb{P} \in \mathcal{Q}$, where $\mathcal{Q}$ denotes the set of all equivalent martingale measures.
[2] ξ^{tr} denotes the transposition of ξ.

Remark. From [Jacod and Shiryaev, 2003] we know the following: From the Doob-Meyer decomposition we use the uniquely defined increasing previsible quadratic variation $(\langle X\rangle_t)_{0\le t\le T}$ with $\langle X\rangle_0 = 0$, such that $(X_t^2 - \langle X\rangle_t)_{0\le t\le T}$ is a local martingale with respect to $\mathbb{P}$.

The discounted **cumulative gains** up to time t are given by

$$G_t(\xi) := \int_0^t \xi_s \cdot dX_s.$$

Remark. Note that $G_t(\xi)$ is a square-integrable $\mathbb{P}$-martingale and null at zero. Obviously, its mean is also zero.

A strategy is called **self-financing**, if $V_t = V_0 + G_t(\xi)$. Since we won't always rely on the self-financing constraint, the value process may deviate from the cumulative gains process. Therefore, we introduce the discounted **cumulative costs** by

$$C_t(\phi) := V_t(\phi) - G_t(\xi) = \phi_t^0 + \xi_t \cdot X_t - \int_0^t \xi_s \cdot dX_s. \tag{2.1.1}$$

$C_t(\phi)$ is adapted and càdlàg by construction.

Remark. A strategy is self-financing if the cumulative costs are constant over time, namely $C_t(\phi) = V_0(\phi)$ for $0 \le t \le T$, which is the initial value to start the strategy ϕ.

Currently we only have that $V_t(\phi)$ is a square-integrable $\mathbb{P}$-martingale, if ϕ is self-financing. We would like to maintain the martingale property without relying on the self-financing constraint.

Definition 2.2 (**mean self-financing**)**.** *A strategy ϕ is called* ***mean self-financing****, if the cost process $C(\phi)$ is a $\mathbb{P}$-martingale.*

Lemma 2.1. *A strategy ϕ is mean self-financing, if and only if V is a square-integrable $\mathbb{P}$-martingale.*

Proof. Using Definition 2.2 and Equation (2.1.1) we immediatelly get the assertion as a consequence of the construction of the stochastic integral. □

We later use Lemma 2.2 to get a unique decomposition of a discounted contingent claim $H \in L^2(\mathcal{F}_T, \mathbb{P})$, namely the *Galtchouk-Kunita-Watanabe decomposition.*

Lemma 2.2. *Let X be a local $\mathbb{P}$-martingale. Define $L^2(X) := \{\xi = (\xi_t)_{0\le t\le T} \mid \xi$ is previsible and $||\xi||_X < \infty\}$ with*

$$||\xi||_X := \mathbb{E}\left(\int_0^T \xi_s^{tr}\, d\langle X\rangle_s\, \xi_s\right)^{\frac{1}{2}}$$

and $\mathcal{M}^2(\mathbb{P}) := \{M = (M_t)_{0\le t\le T} \mid M$ is a martingale and $\mathbb{E}(M_T^2) < \infty\}$. Then the space $\mathcal{I}^2(X) := \left\{ \left(\int_0^t \xi_s \cdot dX_s\right)_{0\le t\le T} \,\middle|\, \xi \in L^2(X) \right\}$ is a stable subspace of $\mathcal{M}^2(\mathbb{P})$.

Proof. Notice, that $\mathcal{I}^2(X)$ is a linear subspace of $\mathcal{M}^2(\mathbb{P})$ and stable under stopping. Further, $L^2(X)$ is a Hilbert space. We can identify $\mathcal{M}^2(\mathbb{P})$ with $L^2(\mathcal{F}_T, \mathbb{P})$, since for any $M_T \in L^2(\mathcal{F}_T, \mathbb{P})$ we have that $M_t := \mathbb{E}(M_T \,|\, \mathcal{F}_t)$ is in $\mathcal{M}^2(\mathbb{P})$.
Let $(Y_T^n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{I}^2(X)$. Then there exists a sequence $(\xi^n)_{n\in\mathbb{N}}$ with $\xi^n \in L^2(X)$, such that

$$Y_T^n = \int_0^T \xi_s^n \cdot dX_s.$$

Assume $(Y_T^n)_{n\in\mathbb{N}}$ converges to some $Y_T \in L^2(\mathcal{F}_T, \mathbb{P})$. Then $\forall\, \epsilon > 0\, \exists\, N \in \mathbb{N}$, such that $\forall\, n, m > N$ we have

$$\mathbb{E}\left((Y_T^n - Y_T^m)^2\right)^{\frac{1}{2}} = \|Y_T^n - Y_T^m\|_2 \leq \|Y_T^n - Y_T\|_2 + \|Y_T^m - Y_T\|_2 < \epsilon.$$

From Itô's isometry we get the equality

$$\|Y_T^n - Y_T^m\|_2 = \mathbb{E}\left(\int_0^T (\xi_s^n - \xi_s^m)^{tr}\, d\langle X\rangle_s\, (\xi_s^n - \xi_s^m)\right)^{\frac{1}{2}} = \|\xi^n - \xi^m\|_X.$$

In total we have $\|\xi^n - \xi^m\|_X < \epsilon$. Thus, $(\xi^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(X)$ with limit $\xi \in L^2(X)$. Set $\hat{Y}_T := \int_0^T \xi_s \cdot dX_s$. We now prove that $Y_T = \hat{Y}_T$ in $L^2(\mathcal{F}_T, \mathbb{P})$. Look at

$$\begin{aligned}\|Y_T - \hat{Y}_T\|_2 &\leq \|Y_T - Y_T^n\|_2 + \|Y_T^n - \hat{Y}_T\|_2 \\ &= \mathbb{E}\left((Y_T^n - Y_T)^2\right)^{\frac{1}{2}} + \mathbb{E}\left(\int_0^t (\xi_s^n - \xi_s)^{tr}\, d\langle X\rangle_s\, (\xi_s^n - \xi_s)\right)^{\frac{1}{2}}.\end{aligned}$$

The first part tends to zero by assumption and the second part tends also to zero, since ξ^n tends to ξ. Hence, we have $Y_T = \hat{Y}_T$ in $L^2(\mathcal{F}_T, \mathbb{P})$ and $\mathcal{I}^2(X)$ is a closed subspace of $\mathcal{M}^2(\mathbb{P})$. We even have that $\mathcal{I}^2(X)$ is a stable subspace. □

Since we are in an incomplete market, there are non-attainable, discounted contingent claims H. Thus, by definition we cannot rely on the terminal condition $V_T(\phi) = H$ $\mathbb{P}$-a.s. and on the self-financing constraint at the *same* time. Hence, there are two quadratic approaches. One relies only the terminal condition and tries to minimize the risk along the way. The other uses self-financing strategies and tries to approximate the terminal value.

2.1.1 Terminal constraint

Let $H \in L^2(\mathcal{F}_T, \mathbb{P})$ be a square-integrable random variable. Since we rely only on the terminal condition, we call a strategy ϕ H-**admissible**, if $V_T(\phi) = H$ $\mathbb{P}$-a.s..

An intuitive approach is to minimize the **variance** of the terminal cost $C_T(\phi)$ of an H-admissible strategy ϕ. Since,

$$C_T(\phi) = V_T(\phi) - G_T(\xi) = H - \int_0^T \xi_s \cdot dX_s,$$

we have $\mathbb{E}(C_T(\phi)) = \mathbb{E}(H)$. Thus, the minimization problem is given by

$$\min_\phi \mathbb{E}\left((C_T(\phi) - \mathbb{E}(H))^2\right), \tag{2.1.2}$$

where ϕ runs over all H-admissible strategies. To solve this minimization problem, we use a unique decomposition of H.

By Lemma 2.2 any $H \in L^2(\mathcal{F}_T, \mathbb{P})$ can be uniquely decomposed into

$$H = \mathbb{E}(H) + \int_0^T \xi_s^* \cdot dX_s + L_T^* \quad \mathbb{P}\text{-a.s.}, \tag{2.1.3}$$

where $\xi^* \in L^2(X)$ and $L^* = (L_t^*)_{0\leq t\leq T} \in \mathcal{M}^2(\mathbb{P})$, with $\mathbb{E}(L_0^*) = 0$, is **strongly orthogonal** to $\mathcal{I}^2(X)$. This means, that

$$\left(L_t^* \int_0^t \xi_s^* \cdot dX_s\right)_{0\leq t\leq T}$$

is a zero mean $\mathbb{P}$-martingale. The unique decomposition (2.1.3) is the **Galtchouk-Kunita-Watanabe decomposition** of the claim H.

Remark. In the literature the decomposition can also be found as

$$H = H_0 + \int_0^T \xi_s^* \cdot dX_s + \bar{L}_T^* \quad \mathbb{P}\text{-a.s.}, \tag{2.1.4}$$

where $H_0 := \mathbb{E}(H \,|\, \mathcal{F}_0) \in L^2(\mathcal{F}_0, \mathbb{P})$, $\xi^* \in L^2(X)$ and $\bar{L}^* \in \mathcal{M}_0^2(\mathbb{P})$[3] is strongly orthogonal to $\mathcal{I}^2(X)$. The connection to our introduced decomposition (2.1.3) can be seen by

$$H = \underbrace{\mathbb{E}(H) + L_0^*}_{=H_0} + \int_0^T \xi_s^* \cdot dX_s + \underbrace{L_T^* - L_0^*}_{=\bar{L}_T^*} \quad \mathbb{P}\text{-a.s.},$$

where one shifts the initial random value L_0^* to the $\mathcal{F}_0$-measurable random variable H_0.

Using (2.1.3) we directly get a solution of (2.1.2).

[3]This means that $\bar{L}_0^* = 0$ $\mathbb{P}$-a.s..

Theorem 2.3. *The minimum of (2.1.2) is attained, if and only $\xi = \xi^*$. Its minimal variance is then given by $\mathbb{E}((L_T^*)^2)$.*

Proof. Let $\phi = (\phi^0, \xi)$ be an H-admissible strategy. Then its costs can be written as

$$C_T(\phi) = H - \int_0^T \xi_s \cdot dX_s = \mathbb{E}(H) + \int_0^T (\xi_s^* - \xi_s) \cdot dX_s + L_T^*.$$

Since L_T^* is strongly orthogonal to the stochastic integral we get

$$\mathbb{E}\left((C_T(\phi) - \mathbb{E}(H))^2\right) = \mathbb{E}\left(\left(\int_0^T (\xi_s^* - \xi_s) \cdot dX_s\right)^2\right) + \mathbb{E}\left((L_T^*)^2\right).$$

By Itô's Isometry it holds

$$\mathbb{E}\left(\left(\int_0^T (\xi_s^* - \xi_s) \cdot dX_s\right)^2\right) = \mathbb{E}\left(\int_0^T (\xi_s^* - \xi_s)^{tr} \, d\langle X\rangle_s \, (\xi_s^* - \xi_s)\right).$$

The last expression is equal to zero if and only if $\xi = \xi^*$ in the $L^2(X)$ sense. □

By the above theorem only ξ^* is fully specified. $(\phi^0)^*$ is only specified at maturity T to fulfill the H-admissibility condition, namely $(\phi_T^0)^* = H - \xi_T^* \cdot X_T$. An example of an optimal strategy would be to choose a self-financing strategy ϕ up to time $T-$ and adapt it with the admissibility condition at time T. Choose $C_t(\phi) = \mathbb{E}(H)$ for $0 \le t < T$. By rearranging Equation (2.1.1) we get

$$(\phi_t^0)^* = \mathbb{E}(H) - \xi_t^* \cdot X_t + \int_0^t \xi_s^* \cdot dX_s \quad (0 \le t < T).$$

At the terminal date T we have $(\phi_T^0)^* = H - \xi_T^* \cdot X_T$ and $C_T(\phi) = \mathbb{E}(H) + L_T^*$. Obviously, we would like to have a better criterion to be able to determine ϕ^0 more precisely. Therefore, we need to measure the risk differently.

Definition 2.3 (risk process). *The* ***risk process*** *is defined by*

$$R_t(\phi) := \mathbb{E}\left((C_T(\phi) - C_t(\phi))^2 \,\middle|\, \mathcal{F}_t\right) \quad (0 \le t \le T), \tag{2.1.5}$$

where we choose a càdlàg version.

Remark. The risk process can be interpreted as the conditional mean squared error process. In the special case, where ϕ is mean self-financing, Equation (2.1.5) is a mean variance criterion, since it simplifies to

$$R_t(\phi) = \mathbb{V}\mathrm{ar}(C_T(\phi) \,|\, \mathcal{F}_t) \quad (0 \le t \le T).$$

We need to compare strategies in a fair way.

Definition 2.4 (**admissible continuation, risk-minimizing**). *Let ϕ and ϕ' be strategies. $\phi' = ((\phi^0)', \xi')$ is called **admissible continuation** of ϕ from t on, if*

$$V_T(\phi') = V_T(\phi) \ \mathbb{P}\text{-}a.s., \ (\phi^0_s)' = \phi^0_s \text{ for } 0 \leq s < t \text{ and } \xi'_s = \xi_s \text{ for } 0 \leq s \leq t.$$

*ϕ is called **risk-minimizing**, if $\forall\, 0 \leq t \leq T$*

$$R_t(\phi) \leq R_t(\phi') \quad \mathbb{P}\text{-}a.s.. \tag{2.1.6}$$

Note, that for an admissible continuation we did not ask for H-admissibility. We only minimize over strategies with the same terminal value.

Remark. Observe the following:

(i) If ϕ is self-financing, then it is risk-minimizing. From Equation (2.1.1) we get that for a self-financing strategy we have $C_t(\phi) = V_0(\phi)$. Therefore, by definition $R_t(\phi) = 0$.

(ii) Suppose $\phi = (\phi^0, \xi)$ is risk-minimizing and H-admissible, then ϕ solves (2.1.2). The Equations (2.1.5) and (2.1.6) for $t = 0$ now imply that ϕ minimizes

$$\begin{aligned}\mathbb{E}\left((C_T(\phi) - C_0(\phi))^2\right) &= \mathbb{V}\text{ar}(C_T(\phi)) + \mathbb{E}\left(C_T(\phi) - C_0(\phi)\right)^2 \\ &= \mathbb{E}\left((C_T(\phi) - \mathbb{E}(H))^2\right) + \left(\mathbb{E}(C_T(\phi)) - C_0(\phi)\right)^2,\end{aligned}$$

where we used the H-admissibility of ϕ. Since ϕ is risk-minimizing, the left hand side is minimal and therefore also the right hand side is minimal. Thus, ϕ solves (2.1.2) and $\xi = \xi^*$.
Further, $\left(\mathbb{E}(C_T(\phi)) - C_0(\phi)\right)^2$ is minimal. We know $\mathbb{E}(C_T(\phi)) = \mathbb{E}(H)$ and $C_0(\phi) = V_0(\phi) = \phi^0_0 + \xi^*_0 \cdot X_0$. Minimizing $\left(\mathbb{E}(C_T(\phi)) - C_0(\phi)\right)^2$, we set

$$\mathbb{E}(C_T(\phi)) \overset{!}{=} C_0(\phi) \Leftrightarrow \mathbb{E}(H) \overset{!}{=} \phi^0_0 + \xi^*_0 \cdot X_0.$$

Thus, for a risk-minimizing strategy we additionally get the initial cash amount $\phi^0_0 = \mathbb{E}(H) - \xi^*_0 \cdot X_0$.

Lemma 2.4. *Any risk-minimizing H-admissible strategy ϕ is also mean self-financing.*

Proof. We need to show, that $C(\phi) = (C_t(\phi))_{0 \leq t \leq T}$ is a $\mathbb{P}$-martingale. Let ϕ be risk-minimizing and H-admissible. Fix $t_0 \in [0, T)$ and define ϕ' by $\xi' := \xi$ and $(\phi^0_t)' := \phi^0_t$ for $t \in [0, t_0)$ and

$$(\phi^0_t)' := \hat{C}_t(\phi) + \int_0^t \xi_s \cdot dX_s - \xi_t \cdot X_t \quad \text{for } t \in [t_0, T],$$

where we choose a càdlàg version of $\hat{C}_t(\phi) = \mathbb{E}(C_T(\phi)\,|\,\mathcal{F}_t)$. $\phi' = ((\phi^0)', \xi)$ is an admissible continuation of ϕ, since

$$\begin{aligned} V_T(\phi') = (\phi_T^0)' + \xi_T \cdot X_T = \hat{C}_T(\phi) + \int_0^T \xi_s \cdot dX_s - \xi_T \cdot X_T + \xi_T \cdot X_T \\ = C_T(\phi) + \int_0^T \xi_s = V_T = H \quad \mathbb{P}\text{-a.s.}. \end{aligned}$$

Further, $C_T(\phi') = C_T(\phi)$, since

$$C_t(\phi') = V_t(\phi') - \int_0^t \xi_s \cdot dX_s = (\phi_t^0)' + \xi_t \cdot X_t - \int_0^t \xi_s \cdot dX_s = \hat{C}_t(\phi) = \mathbb{E}(C_T(\phi)\,|\,\mathcal{F}_t).$$

Hence, $C_t(\phi') = \mathbb{E}(C_T(\phi)\,|\,\mathcal{F}_t) = \mathbb{E}(C_T(\phi')\,|\,\mathcal{F}_t)$ is a martingale for $t \in [t_0, T]$. Together with

$$C_T(\phi) - C_{t_0}(\phi) = C_T(\phi') - C_{t_0}(\phi') + \mathbb{E}(C_T(\phi')\,|\,\mathcal{F}_{t_0}) - C_{t_0}(\phi)$$

we get

$$\begin{aligned} R_{t_0}(\phi) &= \mathbb{E}\left((C_T(\phi) - C_{t_0}(\phi))^2 \,\middle|\, \mathcal{F}_{t_0}\right) \\ &= \mathbb{E}\left(\left(C_T(\phi') - C_{t_0}(\phi') + \mathbb{E}(C_T(\phi')\,|\,\mathcal{F}_{t_0}) - C_{t_0}(\phi)\right)^2 \,\middle|\, \mathcal{F}_{t_0}\right) \\ &= R_{t_0}(\phi') + \left(\mathbb{E}(C_T(\phi')\,|\,\mathcal{F}_{t_0}) - C_{t_0}(\phi)\right)^2, \end{aligned}$$

where the middle term vanished, since $\mathbb{E}(C_T(\phi') - C_{t_0}(\phi')\,|\,\mathcal{F}_{t_0}) = 0$. Since ϕ is risk-minimizing, the right hand side must be minimal. Therefore, we conclude that

$$C_{t_0}(\phi) = \mathbb{E}(C_T(\phi')\,|\,\mathcal{F}_{t_0}) = \mathbb{E}(C_T(\phi)\,|\,\mathcal{F}_{t_0}) \quad \mathbb{P}\text{-a.s.}$$

and since t_0 was fixed arbitrarily, it follows that $C(\phi)$ is a $\mathbb{P}$-martingale. □

Remark. In the above proof we *did not* use that X is a local $\mathbb{P}$-martingale. Thus, Lemma 2.4 will specifically come in handy later in the general case, where X is only a semimartingale[4].

Definition 2.5 (intrinsic risk process). *The* ***intrinsic risk process*** *of a claim H is denoted by $R^* = (R_t^*)_{0 \le t \le T}$, where we choose a càdlàg version of*

$$R_t^* := \mathbb{E}\left((L_T^* - L_t^*)^2 \,\middle|\, \mathcal{F}_t\right) \quad (0 \le t \le T). \tag{2.1.7}$$

For the next theorem let V^* be a square-integrable martingale defined by

$$V_t^* := \mathbb{E}(H\,|\,\mathcal{F}_t) \quad (0 \le t \le T),$$

where we choose a càdlàg version.

[4]Compare the proof of Lemma 2.8.

Theorem 2.5. *Suppose X is a local $\mathbb{P}$-martingale. Then every claim $H \in L^2(\mathcal{F}_T, \mathbb{P})$ has a unique H-admissible, risk-minimizing strategy ϕ^* given by*

$$\phi_t^* = (V_t^* - \xi_t^* \cdot X_t,\, \xi_t^*) \quad (0 \leq t \leq T),$$

and the remaining risk is R_t^* $\mathbb{P}$-a.s. for every $0 \leq t \leq T$. Its cost process is given by $C_t(\phi^*) = \mathbb{E}(H) + L_t^*$.

Proof. First, we check the admissibility condition. We have

$$V_T(\phi^*) = V_T^* - \xi_T^* \cdot X_T + \xi_T^* \cdot X_T = V_T^* = H \quad \mathbb{P}\text{-a.s.}.$$

Second, we check the cost process $C_t(\phi^*)$. The Galtchouk-Kunita-Watanabe decomposition of V^* yields

$$\begin{aligned} V_t^* = \mathbb{E}(H \,|\, \mathcal{F}_t) &= \mathbb{E}\left(\mathbb{E}(H) + \int_0^T \xi_s^* \cdot dX_s + L_T^* \,\middle|\, \mathcal{F}_t\right) \\ &= \mathbb{E}(H) + \int_0^t \xi_s^* \cdot dX_s + L_t^*. \end{aligned}$$

By Lemma 2.4 we know, that ϕ^* is also mean self-financing, which by Lemma 2.1 is equivalent to $V(\phi^*)$ beeing a square-integrable $\mathbb{P}$-martingale. Thus, for the cost process of ϕ^* we get with $V_t(\phi^*) = \mathbb{E}(V_T(\phi^*) \,|\, \mathcal{F}_t) = \mathbb{E}(V_T^* \,|\, \mathcal{F}_t) = V_t^*$

$$C_t(\phi^*) = V_t^* - \int_0^T \xi_s^* \cdot dX_s = \mathbb{E}(H) + L_t^*.$$

Third, we check the minimality of ϕ^*. Fix $t \in [0, T)$ and let ϕ' be an admissible continuation of ϕ^*. Looking at the cost process of ϕ', we have

$$\begin{aligned} C_T(\phi') - C_t(\phi') &= V_T(\phi') - \int_0^T \xi_s' \cdot dX_s - V_t(\phi') + \int_0^t \xi_s' \cdot dX_s \\ &= H - \int_t^T \xi_s' \cdot dX_s - V_t(\phi') \\ &= \mathbb{E}(H) + \int_0^T \xi_s^* \cdot dX_s + L_T^* - \int_t^T \xi_s' \cdot dX_s - V_t(\phi') \\ &= V_t^* - V_t(\phi') + L_T^* - L_t^* + \int_t^T (\xi_s^* - \xi_s') \cdot dX_s. \end{aligned}$$

Using the strong orthoganility of L^H and the stochastic integral we get

$$\begin{aligned} R_t(\phi') &= \mathbb{E}\left((C_T(\phi') - C_t(\phi'))^2 \,\middle|\, \mathcal{F}_t\right) \\ &= \mathbb{E}\left((V_t^* - V_t(\phi'))^2 \,\middle|\, \mathcal{F}_t\right) + R_t^* + \mathbb{E}\left(\int_t^T (\xi_s^* - \xi_s')^{tr}\, d\langle X\rangle_s\, (\xi_s^* - \xi_s') \,\middle|\, \mathcal{F}_t\right). \end{aligned}$$

If ϕ^* is risk-minimizing, only R_t^* remains.
At last, we check the uniqueness of ϕ^*. Assume $\phi = (\phi^0, \xi)$ is another H-admissible, risk-minimizing strategy. Then ϕ solves also (2.1.2), which yields $\xi = \xi^*$. Again, by the Lemmata 2.4 and 2.1 we have that $V(\phi)$ is a square-integrable $\mathbb{P}$-martingale. Since ϕ is H-admissible, we further have $V_T(\phi) = V_T(\phi^*)$, which yields

$$V_t(\phi) = \mathbb{E}(V_T(\phi) \,|\, \mathcal{F}_t) = \mathbb{E}(V_T(\phi^*) \,|\, \mathcal{F}_t) = \mathbb{E}(V_T^* \,|\, \mathcal{F}_t) = V_t^*.$$

Thus, $\phi_t^0 = V_t(\phi) - \xi_t \cdot X_t = V_t^* - \xi_t^* \cdot X_t$. □

In the next lemma we will see a summary of the obtained results in a complete market. More specifically, we assume that H is attainable. Recall, that H is attainable, if $V_t(\phi) = V_0(\phi) + \int_0^t \xi_s \cdot dX_s$ with terminal value $V_T(\phi) = H$ $\mathbb{P}$-a.s..

Lemma 2.6. *The following statements are equivalent:*

(i) *H is attainable and $H = \mathbb{E}(H) + \int_0^T \xi_s^* \cdot dX_s$ $\mathbb{P}$-a.s.,*

(ii) *the risk-minimizing strategy is self-financing,*

(iii) *The intrinsic risk process of H is zero.*

Proof. The equivalences follow immediately by the respective definitions and the above results. □

2.1.2 Self-financing constraint

An alternative approach for hedging an unattainable claim H is to rely on the self-financing constraint. The error at the terminal date T is then given by

$$H - V_T(\phi) = H - V_0(\phi) - \int_0^T \xi_s \cdot dX_s.$$

Since ϕ is self-financing, the strategy is uniquely determined by the choice of $(V_0, \xi) \in \mathbb{R} \times L^2(X)$ (compare Equation (1.1.2)). Then the minimization problem is given by

$$\min_{(V_0,\xi)\in\mathbb{R}\times L^2(X)} \mathbb{E}\left(\left(H - V_0 - \int_0^T \xi_s \cdot dX_s\right)^2\right). \tag{2.1.8}$$

To find its solution we have to project $H \in L^2(\mathcal{F}_T, \mathbb{P})$ onto the linear space spanned by $\mathbb{R}$ and

$$\left\{ \int_0^T \xi_s \cdot dX_s \,\middle|\, \xi \in L^2(X) \right\}.$$

Fortunately, since X is a local martingale the stochastic integral is an isometry and hence this linear space is closed in $L^2(\mathbb{P})$. Hence, by Hilbert's Projection Theorem the solution exists and is unique. By using (2.1.3) the solution is directly given by

$$(V_0, \xi) = (\mathbb{E}(H), \xi^*)$$

and the minimal residual risk by optional stopping is$\mathbb{E}((L_T^*)^2) = \mathbb{V}\mathrm{ar}(L_T^*)$, since $L^* \in \mathcal{M}^2(\mathbb{P})$ with $\mathbb{E}(L_0^*) = 0$.

In the general case, where X is a semimartingale, this projection idea will lead to *mean-varinace hedging.* Further, the generalized Galtchouk-Kunita-Watanabe decomposition will play an important role in both quadratic hedging approaches and we will often choose a suitable equivalent local martingale measure to simplify our work.

2.2 Local risk-minimization

The structure of the following section is based on [Schweizer, 1999]. The details are worked out in different references, which are mentioned throughout this section.
Assume, that the adapted, càdlàg, d-dimensional process X is now only a semimartingale with respect to $\mathbb{P}$ and *not* a local $\mathbb{P}$-martingale. As in the martingale case, we would like to find for a claim $H \in L^2(\mathcal{F}_T, \mathbb{P})$ an H-admissible, risk-minimizing strategy ϕ^*. Unfortunately, this is not possible.

Theorem 2.7. *Assume X is not a local $\mathbb{P}$-martingale. Then a claim $H \in L^2(\mathcal{F}_T, \mathbb{P})$ has in general no H-admissible, risk-minimizing strategy ϕ^*.*

Proof. We will prove this theorem by giving a counterexample in the simplified case of a time discrete market model. Then the filtration is given by $\mathcal{F} = (\mathcal{F}_k)_{k=0,1,\ldots,T}$ with $T \in \mathbb{N}$. Let $X = (X_k)_{k=0,1,\ldots,T}$ be a one dimensional, real valued, square-integrable, $\mathcal{F}$-adapted process and assume, that the probability space is finite. We need this assumption to ensure that all integrability conditions are fulfilled.

Assume ϕ^* is an H-admissible, risk-minimizing strategy. Then $C(\phi^*)$ is by Lemma 2.4[5] a $\mathbb{P}$-martingale, which yields

$$\begin{aligned}
R_k(\phi^*) &= \mathbb{E}\left((C_T(\phi^*) - C_k(\phi^*))^2 \,\middle|\, \mathcal{F}_k \right) \\
&= \mathbb{V}\mathrm{ar}(C_T(\phi^*) - C_k(\phi^*) \,|\, \mathcal{F}_k) + \mathbb{E}\left(C_T(\phi^*) - C_k(\phi^*) \,|\, \mathcal{F}_k \right)^2 \\
&= \mathbb{V}\mathrm{ar}(C_T(\phi^*) \,|\, \mathcal{F}_k) \\
&= \mathbb{V}\mathrm{ar}\left(V_T(\phi^*) - \sum_{i=1}^{T} \xi_i^* \Delta X_i \,\middle|\, \mathcal{F}_k \right) \\
&= \mathbb{V}\mathrm{ar}\left(H - \sum_{i=k+1}^{T} \xi_i^* \Delta X_i \,\middle|\, \mathcal{F}_k \right),
\end{aligned}$$

[5] Compare the remark afterwards.

where $\Delta X_i = X_i - X_{i-1}$. Using $C_k(\phi^*) = \mathbb{E}(C_T(\phi^*) \mid \mathcal{F}_k) = \mathbb{E}\left(H - \sum_{i=1}^T \xi_i^* \Delta X_i \,\middle|\, \mathcal{F}_k\right)$, we further have

$$
\begin{aligned}
(\phi_k^0)^* + \xi_k^* X_k = V_k(\phi^*) &= C_k(\phi^*) + \sum_{i=1}^k \xi_i^* \Delta X_i \\
&= \mathbb{E}\left(H - \sum_{i=1}^T \xi_i^* \Delta X_i \,\middle|\, \mathcal{F}_k\right) + \sum_{i=1}^k \xi_i^* \Delta X_i.
\end{aligned}
$$

Thus, ϕ^* is uniquely determined by ξ^*. Since ϕ^* is risk-minimizing, it holds for any H-admissible, mean self-financing strategy ϕ

$$
\mathbb{V}\mathrm{ar}\left(H - \sum_{i=k+1}^T \xi_i^* \Delta X_i \,\middle|\, \mathcal{F}_k\right) = R_k(\phi^*) \leq R_k(\phi) = \mathbb{V}\mathrm{ar}\left(H - \sum_{i=k+1}^T \xi_i \Delta X_i \,\middle|\, \mathcal{F}_k\right).
$$

By a backward recursion argument, we have to minimize

$$
\mathbb{V}\mathrm{ar}\left(H - \xi_{k+1} \Delta X_{k+1} - \sum_{i=k+2}^T \xi_i \Delta X_i \,\middle|\, \mathcal{F}_k\right)
$$

with respect to ξ_{k+1}, which is $\mathcal{F}_k$-measurable. The minimum is attained, if and only if

$$
\mathbb{C}\mathrm{ov}\left(H - \xi_{k+1} \Delta X_{k+1} - \sum_{i=k+2}^T \xi_i \Delta X_i, \Delta X_{k+1} \,\middle|\, \mathcal{F}_k\right) = 0.
$$

Thus, ξ_{k+1}^* is uniquely determined by

$$
\xi_{k+1}^* = \frac{\mathbb{C}\mathrm{ov}\left(H - \sum_{i=k+2}^T \xi_i^* \Delta X_i, \Delta X_{k+1} \,\middle|\, \mathcal{F}_k\right)}{\mathbb{V}\mathrm{ar}(\Delta X_{k+1} \mid \mathcal{F}_k)}. \tag{2.2.1}
$$

To give a counterexample, choose $T = 2$ and let X be a random walk with $X_0 = 0$ and i.i.d. increments, which only take the three values $-1, 0, 1$ with probabilities $\mathbb{P}(\Delta X_i = -1) = 1/2$, $\mathbb{P}(\Delta X_i = 0) = 1/4$ and $\mathbb{P}(\Delta X_i = 1) = 1/4$ for $i = 1, 2$. Let $H = X_2^2$. The information flow is given by the canonical filtration $\mathcal{F}^X$ of X. We have $\mathcal{F}_0^X = \{\emptyset, \Omega\}$, $\mathcal{F}_1^X = \sigma(\Delta X_1)$ and $\mathcal{F}_2^X = \sigma(\Delta X_1, \Delta X_2)$, where $\Omega = \{-1, 0, 1\}^2$ and $\mathcal{A} = \mathcal{P}(\Omega)$. We have three possible choices of ξ_2 depending on X_1, which we abbriviate by $\xi_2(-1)$, $\xi_2(0)$ and $\xi_2(1)$.

Assume there exists an H-admissible, risk-minimizing strategy ϕ^*. Then ξ^* is given by (2.2.1) and we get $\xi_2^*(-1) = 23/11$, $\xi_2^*(0) = -1/11$, $\xi_2^*(+1) = 21/11$ and $\xi_1^* = -1/11$. In total we get

$$
R_0(\phi^*) = \frac{24}{66}.
$$

On the other hand, for any H-admissible, mean self-financing strategy ϕ we can view $R_0(\phi)$ as a function of $\xi_1, \xi_2(-1), \xi_2(0)$ and $\xi_2(1)$. Minimizing this function in these four variables we get $\xi_1' = -1/11, \xi_2'(-1) = -71/33$, $\xi_2'(0) = 5/33$ and $\xi_2'(1) = 59/33$ and in total

$$R_0(\phi') = \frac{23}{66} < R_0(\phi^*).$$

Thus, there exists no H-admissible strategy ϕ^*, which is risk-minimizing. □

Remark. Intuitively, we have a compatibility problem. Since we only look at admissible continuations of ϕ from t on, the optimal continuation is only optimal at time t. Hence, for $s < t$, the s-optimal strategy may be different from the t-optimal strategy. But since $(s,T] \supset (t,T]$, the s-optimal strategy already defines the strategy on $(t,T]$, which leads to compatibility problems. In short, the admissible continuation criterion is not time consistent. The extraordinary result on the other hand is that, if X is a local $\mathbb{P}$-martingale, we do not have this compatibility problem and we get a unique H-admissible, risk minimizing strategy ϕ^* (compare Theorem 2.5).

Since we assumed, that the adapted, càdlàg process X is a semimartingale with respect to $\mathbb{P}$, we know from [Protter, 2005] that X admits the decomposition

$$X_t = X_0 + M_t + A_t \quad (0 \le t \le T), \tag{2.2.2}$$

where $M \in \mathcal{M}^2_{0,\text{loc}}(\mathbb{P})$ is a locally square-integrable local $\mathbb{P}$-martingale with $M_0 = 0$ and A is a finite variation process with $A_0 = 0$.

Definition 2.6 (**L^2-strategy**)**.** *Let X be a semimartingale satisfying* (2.2.2)*. Define $\Xi := \{\xi = (\xi_t)_{0\le t\le T} \,|\, \xi$ is previsible and $\|\xi\|_\Xi < \infty\}$, where*

$$\|\xi\|_\Xi := \mathbb{E}\left(\int_0^T \xi_s^{tr}\, d\langle M\rangle_s\, \xi_s + \left(\int_0^T |\xi_s^{tr}\, dA_s|\right)^2\right)^{\frac{1}{2}}.$$

*Then $\phi = (\phi^0, \xi)$ is called L^2-**strategy** if ϕ^0 is $\mathcal{F}$-adapted and $\xi \in \Xi$, such that the value process $V(\phi)$ is càdlàg and square-integrable.*

Remark. In case of X beeing a local $\mathbb{P}$-martingale we have $A \equiv 0$. This yields $\Xi = L^2(X)$ and the L^2-strategy coincides with the strategy of Definition 2.1.

2.2.1 Small pertubations and R-quotients

Let us now restrict ourselfs to the case $d = 1$ to simplify our notation. In the case $d > 1$ analogous results can be obtained. The idea of the following definition is a variational argument. If we change the optimal strategy in a small way, we should have (asymptotically) an increased risk.

Definition 2.7 (**small pertubation**). *Let $\Delta = (\varepsilon, \delta)$ be an L^2-strategy with $\varepsilon_T = \delta_T = 0$. Further, suppose that δ is bounded and $\int_0^T |\delta_s| \, |dA|_s$ is uniformly bounded in ω and t. Then Δ is called a* ***small pertubation*** *and on $(s,t] \subset [0,T]$ it is defined as*

$$\Delta|_{(s,t]} := \left(\varepsilon|_{[s,t)}, \delta|_{(s,t]}\right),$$

with

$$\varepsilon|_{[s,t)}(\omega, u) := \varepsilon_u(\omega)\mathbb{I}_{[s,t)}(u), \quad \delta|_{(s,t]}(\omega, u) := \delta_u(\omega)\mathbb{I}_{(s,t]}(u).$$

Remark. The definition of $\Delta|_{(s,t]}$ reflects the fact, that ε is adapted and δ is previsible.

Remark. Since $X = X_0 + M + A$, where M is the unprevisible martingale part and A the drift, $\int_0^T \xi_s \, dA_s$ could be interpreted as the systematic gains of the strategy Δ. Its assumption of bounded total variation means, that these systematic gains are limited and in this sense are small enough.
The condition $\varepsilon_T = \delta_T = 0$ implies $V_T(\Delta) = 0$ $\mathbb{P}$-a.s.. Therefore, we have $V_T(\phi + \Delta) = V_T(\phi) = H$ $\mathbb{P}$-a.s., if ϕ is H-admissible. Hence, $\phi + \Delta$ is H-admissible.

Definition 2.8 (**local risk-minimization**). *Let ϕ be an L^2-strategy and Δ be a small pertubation. For a partition $\pi = \{t_0, t_1, \dots, t_N\}$ of $[0,T]$ with $0 = t_0 < t_1 < \cdots < t_N = T$ we define the* ***R-quotient*** *as*

$$r^\pi(\phi, \Delta)(\omega, t) := \sum_{i=0}^{N-1} \frac{R_{t_i}\left(\phi + \Delta|_{(t_i, t_{i+1}]}\right) - R_{t_i}(\phi)}{\mathbb{E}(\langle M \rangle_{t_{i+1}} - \langle M \rangle_{t_i} \mid \mathcal{F}_{t_i})}(\omega)\mathbb{I}_{(t_i, t_{i+1}]}(t).$$

Suppose it holds for every small pertubation Δ and every increasing sequence of partitions $(\pi_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} |\pi_n| = 0$, where $|\pi_n| := \max_{i\in\{0,1,\dots,N-1\}}(t_{i+1} - t_i)$, that

$$\liminf_{n\to\infty} r^{\pi_n}(\phi, \Delta) \geq 0 \quad (\mathbb{P} \otimes \langle M \rangle)\text{-a.e. on } \Omega \times [0,T]. \tag{2.2.3}$$

Then ϕ is called ***locally risk-minimizing***.

Remark. Loosely speaking, $r^\pi(\phi, \Delta)$ are the directional derivatives on the respective time scale. If we assume that $\phi' := \phi + \Delta$ is an admissible continuation from t on, then (2.1.6) reads for all $0 \leq t \leq T$ as

$$R_t(\phi + \Delta) - R_t(\phi) \geq 0 \quad \mathbb{P}\text{-a.s..}$$

Thus, Equation (2.2.3) is the infintesimal analogon of (2.1.6).

The following result shows, that a local version of Lemma 2.4 holds true.

Lemma 2.8. *Suppose $d = 1$ and X satisfies the decomposition* (2.2.2). *Assume that $\langle M \rangle$ is $\mathbb{P}$-a.s. strictly increasing. Then a locally risk-minimizing L^2-strategy is mean self-financing.*

Proof. The proof is based on [Schweizer, 1991].
Let ϕ be a locally risk-minimizing L^2-strategy. Choose $t_0 = 0$ and define ϕ' as in Lemma 2.4. Then $\Delta := \phi' - \phi$ is a small pertubation. For the sequence of partitions $(\pi_n)_{n\in\mathbb{N}}$ we choose π_n to be the n-th dyadic partition of $[0,T]$ and define $t_+ := (t + 2^{-n}T) \wedge T$ as the successor of $t \in \pi_n$. Since

$$V_t(\phi + \Delta|_{(t,t_+]}) = V_t(\phi) + (\phi_t^0)' - \phi_t^0 = \phi_t^0 + \xi_t X_t + (\phi_t^0)' - \phi_t^0 = V_t(\phi'),$$

we have $C_t(\phi+\Delta|_{(t,t_+]}) = C_t(\phi')$. Further, $\delta_T = \varepsilon_T = 0$ yields $V_T(\phi+\Delta|_{(t,t_+]}) = V_T(\phi)$ and since $V_T(\phi) = V_T(\phi')$ we get $C_T(\phi + \Delta|_{(t,t_+]}) = C_T(\phi')$. In total we have $\forall\, n \in \mathbb{N}$ and $t \in \pi_n$

$$C_T(\phi + \Delta|_{(t,t_+]}) - C_t(\phi + \Delta|_{(t,t_+]}) = C_T(\phi') - C_t(\phi').$$

Hence,

$$R_t(\phi + \Delta|_{(t,t_+]}) = R_t(\phi').$$

From the proof of Lemma 2.4 we have for $T \geq t \geq t_0 = 0$

$$R_t(\phi) - R_t(\phi') = \left(\mathbb{E}(C_T(\phi)\,|\,\mathcal{F}_t) - C_t(\phi)\right)^2.$$

Thus,

$$R_t(\phi + \Delta|_{(t,t_+]}) - R_t(\phi) = R_t(\phi') - R_t(\phi) = -\left(\mathbb{E}(C_T(\phi)\,|\,\mathcal{F}_t) - C_t(\phi)\right)^2$$

and we obtain

$$r^{\pi_n}(\phi,\Delta) = -\sum_{t\in\pi_n} \frac{\left(\mathbb{E}(C_T(\phi)\,|\,\mathcal{F}_t) - C_t(\phi)\right)^2}{\mathbb{E}(\langle M\rangle_{t_+} - \langle M\rangle_t\,|\,\mathcal{F}_t)} \mathbb{I}_{(t,t_+]}. \tag{2.2.4}$$

If $\mathbb{E}(C_T(\phi)\,|\,\mathcal{F}_t) = C_t(\phi)$ for $0 \leq t \leq T$, we are done. Otherwise there exists a dyadic rational q and a set B of positive probability, such that

$$\mathbb{E}(C_T(\phi)\,|\,\mathcal{F}_q)(\omega) \neq C_q(\phi)(\omega) \quad \forall\, \omega \in B.$$

By right continuity of $\mathbb{E}(C_T(\phi)\,|\,\mathcal{F}_\cdot)$ and $C_\cdot(\phi)$ we have $\forall\, \omega \in B\, \exists$ constants $c(\omega) > 0, \beta(\omega) > 0$, such that

$$\left|\mathbb{E}(C_T(\phi)\,|\,\mathcal{F}_t) - C_t(\phi)\right|(\omega) \geq c(\omega) > 0 \quad \text{for any dyadic rational } t \in [q, q + \beta(\omega)].$$

But then (2.2.4) implies for $s \in (q, q+\beta(\omega))$

$$\liminf_{n\to\infty} r^{\pi_n}(\phi,\Delta)(\omega,s) < 0,$$

contradicting our assumption of ϕ beeing locally risk-minimizing. Thus, we have

$$\mathbb{E}(C_T(\phi)\,|\,\mathcal{F}_t) = C_t(\phi) \quad \mathbb{P}\text{-a.s. for every dyadic rational } t$$

and since the dyadic rationals are dense in $[0,T]$, the assertion follows from right continuity. □

Remark. The condition, that $\langle M\rangle$ is $\mathbb{P}$-a.s. strictly increasing is necessary to exclude the possibility of M beeing locally constant. This would lead to problems in the definition of $r^{\pi_n}(\phi,\Delta)$.

Thanks to Lemma 2.8 we can restrict our search for a locally risk-minimizing strategy to mean self-financing strategies. Following [Schweizer, 1991] we will split $r^{\pi}(\phi,\Delta)$ into two terms, where the first one only depends on ξ and δ and the second only on ϕ^0 and ε. The assumptions in the following Lemma 2.9 are needed to ensure that the second term vanishes asymptotically.

Let $H\in L^2(\mathcal{F}_T,\mathbb{P})$ and ϕ be an H-admissible, mean self-financing L^2-strategy. The terminal constraint yields

$$C_T(\phi)=H-\int_0^T \xi_s\,dX_s \quad \mathbb{P}\text{-a.s.}$$

and since $C(\phi)$ is a martingale, we have

$$\phi_t^0=\mathbb{E}(H\,|\,\mathcal{F}_t)-\xi_t X_t-\mathbb{E}\left(\int_t^T \xi_s\,dX_s \,\middle|\, \mathcal{F}_t\right)$$

and ϕ is uniquely determined by ξ. Thus, we have to deal with one dimension less and we abbreviate $C_t(\xi):=C_t(\phi)$ and $R_t(\xi):=R_t(\phi)$. As usual, let Δ be a small pertubation and π be a partition of $[0,T]$. Then for $t_i\in\pi$ we assume, that $\xi+\delta|_{(t_i,t_{i+1}]}$ is H-admissible and mean self-financing. $\phi+\Delta|_{(t_i,t_{i+1}]}$ is H-admissible, but may not be mean self-financing due to the merely adapted component $\varepsilon|_{[t_i,t_{i+1})}$. With $V_T(\phi+\Delta|_{(t_i,t_{i+1}]})=V_T(\phi)=H$ $\mathbb{P}$-a.s. we have

$$\begin{aligned}
C_T(\phi+\Delta|_{(t_i,t_{i+1}]}) &= V_T(\phi+\Delta|_{(t_i,t_{i+1}]})-\int_0^T \xi_s\,dX_s-\int_{t_i}^{t_{i+1}}\delta_s\,dX_s\\
&= H-\int_0^T \xi_s\,dX_s-\int_{t_i}^{t_{i+1}}\delta_s\,dX_s\\
&= C_T(\phi)-\int_{t_i}^{t_{i+1}}\delta_s\,dX_s\\
&= C_T(\xi+\delta|_{(t_i,t_{i+1}]}).
\end{aligned} \tag{2.2.5}$$

Further, it holds

$$\begin{aligned}
C_{t_i}(\phi+\Delta|_{(t_i,t_{i+1}]}) &= V_{t_i}(\phi+\Delta|_{(t_i,t_{i+1}]})-\int_0^{t_i}\xi_s\,dX_s\\
&= V_{t_i}(\phi)+\varepsilon_{t_i}-\int_0^{t_i}\xi_s\,dX_s\\
&= C_{t_i}(\phi)+\varepsilon_{t_i}
\end{aligned}$$

and since $\xi + \delta|_{(t_i,t_{i+1}]}$ is mean self-financing we have

$$\begin{aligned} C_{t_i}(\xi + \delta|_{(t_i,t_{i+1}]}) &= \mathbb{E}(C_T(\xi + \delta|_{(t_i,t_{i+1}]}) \,|\, \mathcal{F}_{t_i}) \\ &= \mathbb{E}\left(C_T(\phi) - \int_{t_i}^{t_{i+1}} \delta_s \, dX_s \,\middle|\, \mathcal{F}_{t_i} \right) \\ &= C_{t_i}(\phi) - \mathbb{E}\left(\int_{t_i}^{t_{i+1}} \delta_s \, dA_s \,\middle|\, \mathcal{F}_{t_i} \right), \end{aligned}$$

where we used the decomposition (2.2.2) of X. This implies

$$C_{t_i}(\phi + \Delta|_{(t_i,t_{i+1}]}) = C_{t_i}(\xi + \delta|_{(t_i,t_{i+1}]}) + \mathbb{E}\left(\int_{t_i}^{t_{i+1}} \delta_s \, dA_s \,\middle|\, \mathcal{F}_{t_i} \right) + \varepsilon_{t_i}. \tag{2.2.6}$$

Now the Equations (2.2.5) and (2.2.6) yield

$$\begin{aligned} R_{t_i}(\phi + \Delta|_{(t_i,t_{i+1}]}) &= \mathbb{E}\left((C_T(\phi + \Delta|_{(t_i,t_{i+1}]}) - C_{t_i}(\phi + \Delta|_{(t_i,t_{i+1}]}))^2 \,\middle|\, \mathcal{F}_{t_i} \right) \\ &= R_{t_i}(\xi + \delta|_{(t_i,t_{i+1}]}) + \left(\mathbb{E}\left(\int_{t_i}^{t_{i+1}} \delta_s \, dA_s \,\middle|\, \mathcal{F}_{t_i} \right) + \varepsilon_{t_i} \right)^2, \end{aligned}$$

where the middle term vanished, since $C_\cdot(\xi + \delta|_{(t_i,t_{i+1}]})$ is a martingale. Summing up we obtain

$$\begin{aligned} &r^\pi(\phi, \Delta) \\ &= \sum_{t_i \in \pi} \frac{R_{t_i}(\xi + \delta|_{(t_i,t_{i+1}]}) - R_{t_i}(\xi)}{\mathbb{E}(\langle M \rangle_{t_{i+1}} - \langle M \rangle_{t_i} \,|\, \mathcal{F}_{t_i})} \mathbb{I}_{(t_i,t_{i+1}]} + \sum_{t_i \in \pi} \frac{\left(\mathbb{E}\left(\int_{t_i}^{t_{i+1}} \delta_s \, dA_s \,\middle|\, \mathcal{F}_{t_i} \right) + \varepsilon_{t_i} \right)^2}{\mathbb{E}(\langle M \rangle_{t_{i+1}} - \langle M \rangle_{t_i} \,|\, \mathcal{F}_{t_i})} \mathbb{I}_{(t_i,t_{i+1}]} \\ &=: r^\pi(\xi, \delta) + \sum_{t_i \in \pi} \frac{\left(\mathbb{E}\left(\int_{t_i}^{t_{i+1}} \delta_s \, dA_s \,\middle|\, \mathcal{F}_{t_i} \right) + \varepsilon_{t_i} \right)^2}{\mathbb{E}(\langle M \rangle_{t_{i+1}} - \langle M \rangle_{t_i} \,|\, \mathcal{F}_{t_i})} \mathbb{I}_{(t_i,t_{i+1}]}. \end{aligned} \tag{2.2.7}$$

Lemma 2.9. *Suppose $d = 1$, X satisfies the decomposition* (2.2.2) *and is $\mathbb{P}$-a.s. continuous at maturity T. Assume that $\langle M \rangle$ is $\mathbb{P}$-a.s. strictly increasing. Further, assume that A is continuous and $A \ll \langle M \rangle$ with a density α satisfying $\mathbb{E}_{\mathbb{P} \otimes \langle M \rangle}(|\alpha| \log^+ |\alpha|) < \infty$. Then an L^2-strategy ϕ is locally risk-minimizing, if and only if ϕ is mean self-financing and*

$$\liminf_{n \to \infty} r^{\pi_n}(\xi, \delta) \geq 0 \quad \mathbb{P} \otimes \langle M \rangle\text{-a.e. on } \Omega \times [0, T], \tag{2.2.8}$$

for every previsible process δ satisfying the same conditions as in Definition 2.7 and every increasing sequence of partitions $(\pi_n)_{n \in \mathbb{N}}$ with $\lim_{n \to \infty} |\pi_n| = 0$.

Proof. Using Lemma 2.8 we can assume that ϕ is mean self-financing. Further, assume that Equation (2.2.8) holds. Then from Equation (2.2.7) it immediately follows that ϕ is locally risk-minimizing. For the converse direction, we choose $\varepsilon_{t_i} = 0$ for all $t_i \in \pi_n$. Then

$$\left(\mathbb{E}\left(\int_{t_i}^{t_{i+1}} \delta_s \, dA_s \,\middle|\, \mathcal{F}_{t_i}\right)\right)^2 \leq \|\delta\|_\infty^2 \mathbb{E}\left((|A|_{t_{i+1}} - |A|_{t_i})^2 \,\middle|\, \mathcal{F}_{t_i}\right)$$

yields

$$\sum_{t_i \in \pi_n} \frac{\mathbb{E}\left(\int_{t_i}^{t_{i+1}} \delta_s \, dA_s \,\middle|\, \mathcal{F}_{t_i}\right)^2}{\mathbb{E}(\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i} \mid \mathcal{F}_{t_i})} \mathbb{I}_{(t_i, t_{i+1}]} \leq \|\delta\|_\infty^2 \sum_{t_i \in \pi_n} \frac{\mathbb{E}\left((|A|_{t_{i+1}} - |A|_{t_i})^2 \,\middle|\, \mathcal{F}_{t_i}\right)}{\mathbb{E}(\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i} \mid \mathcal{F}_{t_i})} \mathbb{I}_{(t_i, t_{i+1}]}.$$

With the technical proof of Proposition 3.1 in [Schweizer, 1990] we get the convergence of the last equation and again with Equation (2.2.7) the converse direction follows. Note, that the stated assumptions on the structure of X, as well as the condition $\mathbb{E}_{\mathbb{P}\otimes\langle M\rangle}(|\alpha| \log^+ |\alpha|) < \infty$ are used in the proof of Proposition 3.1 in [Schweizer, 1990]. □

Remark. In the special case, where $\langle M\rangle = t$ and A is absolutely continuous with respect to the Lebesgue measure and bounded density α, the right hand side of the last equation converges to zero $(\mathbb{P} \otimes \langle M\rangle)$-a.e. and we do not need Proposition 3.1 of [Schweizer, 1990].

Remark. The assumption of X beeing continuous at maturity T and A beeing continuous imply that M does not jump at time T and hence, $\langle M\rangle$ has no mass at T.

Thanks to Lemma 2.9 the locally risk-minimization problem splits into two simpler problems. Namely, we only have to find the optimal ξ component and choose ϕ^0 in such a way that ϕ is mean self-financing. Therefore, we look into the R-quotient of the martingale

$$C_t(\xi + \delta_{(t_i, t_{i+1}]}) = \mathbb{E}\left(C_T(\xi) - \int_{t_i}^{t_{i+1}} \delta_s \, dX_s \,\middle|\, \mathcal{F}_t\right) \quad (0 \leq t \leq T), \tag{2.2.9}$$

as it is done in [Schweizer, 1990]. Since M is a local $\mathbb{P}$-martingale, the Galtchouk-Kunita-Watanabe decomposition of $C_T(\xi)$ with respect to $\mathbb{P}$ yields

$$C_T(\xi) = C_0(\xi) + \int_0^T \mu_s^C \, dM_s + L_T^C \quad \mathbb{P}\text{-a.s.}, \tag{2.2.10}$$

where $\mu^C \in L^2(\Omega \times [0,T], \mathcal{P}, \mathbb{P} \otimes \langle M\rangle)$, with $\mathcal{P}$ denoting the σ-algebra of previsible sets and $L^C \in \mathcal{M}^2(\mathbb{P})$, with $\mathbb{E}(L_0^C) = 0$, is strongly orthogonal to $\mathcal{I}^2(M)$.

Lemma 2.10. *Suppose the same assumptions as in Lemma 2.9 hold on X, $\langle M\rangle$, A and α. Then, we have*[6]

$$\lim_{n\to\infty} r^{\pi_n}(\xi,\delta) = \delta^2 - 2\delta\mu^C \quad (\mathbb{P}\otimes\langle M\rangle)\text{-}a.e.,$$

for every previsible process δ satisfying the same conditions as in Definition 2.7 and every increasing sequence of partitions $(\pi_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty}|\pi_n| = 0$.

Proof. By using (2.2.9) and the decomposition (2.2.2) of X we have

$$\begin{aligned} &C_T(\xi+\delta|_{(t_i,t_{i+1}]}) - C_{t_i}(\xi+\delta|_{(t_i,t_{i+1}]}) \\ &\quad = C_T(\xi) - C_{t_i}(\xi) - \int_{t_i}^{t_{i+1}} \delta_s\,dM_s - \left(\int_{t_i}^{t_{i+1}} \delta_s\,dA_s - \mathbb{E}\left(\int_{t_i}^{t_{i+1}} \delta_s\,dA_s \,\middle|\, \mathcal{F}_{t_i}\right)\right). \end{aligned}$$

It holds $R_t(\xi) = \mathbb{E}\left(\langle C(\xi)\rangle_T - \langle C(\xi)\rangle_t \,|\, \mathcal{F}_t\right)$. Then (2.2.10) and the calculation rules for the quadratic variation yield

$$\begin{aligned} &R_{t_i}(\xi+\delta|_{(t_i,t_{i+1}]}) - R_{t_i}(\xi) \\ &\quad = \mathbb{E}\left(\int_{t_i}^{t_{i+1}} \delta_s^2 - 2\delta_s\mu_s^C\,d\langle M\rangle_s \,\middle|\, \mathcal{F}_{t_i}\right) + \mathbb{V}\mathrm{ar}\left(\int_{t_i}^{t_{i+1}} \delta_s\,dA_s \,\middle|\, \mathcal{F}_{t_i}\right) \\ &\qquad + 2\,\mathbb{C}\mathrm{ov}\left(\int_{t_i}^{t_{i+1}} \delta_s\,dM_s - (C_{t_{i+1}}(\xi) - C_{t_i}(\xi))\ ,\ \int_{t_i}^{t_{i+1}} \delta_s\,dA_s \,\middle|\, \mathcal{F}_{t_i}\right). \end{aligned}$$

Summing up we obtain

$$\begin{aligned} r^{\pi_n}(\xi,\delta) &= \mathbb{E}_{\mathbb{P}\otimes\langle M\rangle}\left(\delta^2 - 2\delta\mu^C \,\middle|\, \mathcal{P}^{\pi_n}\right) \\ &\quad + \sum_{t_i\in\pi_n} \frac{\mathbb{V}\mathrm{ar}\left(\int_{t_i}^{t_{i+1}} \delta_s\,dA_s \,\middle|\, \mathcal{F}_{t_i}\right)}{\mathbb{E}\left(\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i} \,\middle|\, \mathcal{F}_{t_i}\right)} \mathbb{I}_{(t_i,t_{i+1}]} \\ &\quad + 2\sum_{t_i\in\pi_n} \frac{\mathbb{C}\mathrm{ov}\left(\int_{t_i}^{t_{i+1}} \delta_s\,dM_s - (C_{t_{i+1}}(\xi) - C_{t_i}(\xi))\ ,\ \int_{t_i}^{t_{i+1}} \delta_s\,dA_s \,\middle|\, \mathcal{F}_{t_i}\right)}{\mathbb{E}\left(\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i} \,\middle|\, \mathcal{F}_{t_i}\right)} \mathbb{I}_{(t_i,t_{i+1}]}, \end{aligned}$$

where

$$\mathcal{P}^{\pi_n} := \sigma\left(\{B_0\times\{0\}, B_i\times(t_i,t_{i+1}] \,|\, B_0\in\mathcal{F}_0, t_i\in\pi_n \text{ for } i=1,\dots,N-1, B_i\in\mathcal{F}_{t_i}\}\right)$$

denotes the σ-algebra on $\Omega\times[0,T]$. Note, that by the assumptions on π_n it holds

$$\mathcal{P} = \sigma\left(\bigcup_{n=1}^{\infty}\mathcal{P}^{\pi_n}\right).$$

[6] Note that μ^C depends on ξ.

Hence, by Doob's Martingale Convergence Theorem we have

$$\lim_{n\to\infty} \mathbb{E}_{\mathbb{P}\otimes\langle M\rangle}\left(\delta^2 - 2\delta\mu^C \,\middle|\, \mathcal{P}^{\pi_n}\right) = \delta^2 - 2\delta\mu^C \quad (\mathbb{P}\otimes\langle M\rangle)\text{-a.e..}$$

The middle term on the right hand side is bounded by

$$\sum_{t_i\in\pi_n} \frac{\mathbb{V}\mathrm{ar}\left(\int_{t_i}^{t_{i+1}} \delta_s\, dA_s \,\middle|\, \mathcal{F}_{t_i}\right)}{\mathbb{E}\left(\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i} \,\middle|\, \mathcal{F}_{t_i}\right)} \mathbb{I}_{(t_i,t_{i+1}]} \leq \sum_{t_i\in\pi_n} \frac{\mathbb{E}\left(\left(\int_{t_i}^{t_{i+1}} \delta_s\, dA_s\right)^2 \,\middle|\, \mathcal{F}_{t_i}\right)}{\mathbb{E}\left(\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i} \,\middle|\, \mathcal{F}_{t_i}\right)} \mathbb{I}_{(t_i,t_{i+1}]}$$

and with Cauchy-Schwarz and the above inequality we obtain for the last term

$$\begin{aligned}
&\left|\sum_{t_i\in\pi_n} \frac{\mathbb{C}\mathrm{ov}\left(\int_{t_i}^{t_{i+1}} \delta_s\, dM_s - (C_{t_{i+1}}(\xi) - C_{t_i}(\xi))\ ,\ \int_{t_i}^{t_{i+1}} \delta_s\, dA_s \,\middle|\, \mathcal{F}_{t_i}\right)}{\mathbb{E}\left(\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i} \,\middle|\, \mathcal{F}_{t_i}\right)} \mathbb{I}_{(t_i,t_{i+1}]}\right| \\
&\quad \leq \left(\sum_{t_i\in\pi_n} \frac{\mathbb{E}\left(\left(\int_{t_i}^{t_{i+1}} \delta_s\, dA_s\right)^2 \,\middle|\, \mathcal{F}_{t_i}\right)}{\mathbb{E}\left(\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i} \,\middle|\, \mathcal{F}_{t_i}\right)} \mathbb{I}_{(t_i,t_{i+1}]}\right)^{\frac{1}{2}} \\
&\qquad \cdot \left(\sum_{t_i\in\pi_n} \frac{\mathbb{E}\left(\int_{t_i}^{t_{i+1}} \delta_s^2\, d\langle M\rangle_s + (\langle C(\xi)\rangle_{t_{i+1}} - \langle C(\xi)\rangle_{t_i})\right)}{\mathbb{E}\left(\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i} \,\middle|\, \mathcal{F}_{t_i}\right)} \mathbb{I}_{(t_i,t_{i+1}]}\right)^{\frac{1}{2}}.
\end{aligned}$$

But since the second term on the right hand side is a nonnegative $(\mathbb{P}\otimes\langle M\rangle, \mathcal{P}^{\pi_n})$-supermartingale, it is bounded by $\mathcal{O}(n)$ $(\mathbb{P}\otimes\langle M\rangle)$-a.e.. Thus, it remains to prove

$$\lim_{n\to\infty} \sum_{t_i\in\pi_n} \frac{\mathbb{E}\left(\left(\int_{t_i}^{t_{i+1}} \delta_s\, dA_s\right)^2 \,\middle|\, \mathcal{F}_{t_i}\right)}{\mathbb{E}\left(\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i} \,\middle|\, \mathcal{F}_{t_i}\right)} \mathbb{I}_{(t_i,t_{i+1}]} = 0 \quad (\mathbb{P}\otimes\langle M\rangle)\text{-a.e..}$$

For this rather technical calculation we refer the reader again to the proof of Proposition 3.1 in [Schweizer, 1990]. □

In summary, we get a characterization theorem for locally risk-minimizing strategies. Before we formulate the characterization, we introduce a certain structure on X based on [Monat and Stricker, 1995]. Assume that

$$A^i \ll \langle M^i\rangle \text{ with previsible density } \alpha^i = (\alpha^i_t)_{0\leq t\leq T} \text{ for } i = 1,\dots,d. \tag{2.2.11}$$

Fix a previsible integrable increasing càdlàg process W with $W_0 = 0$, such that $\langle M^i\rangle \ll W$ for all $i = 0,\dots,d$.[7] By the Kunita-Watanabe Inequality we have that $\langle M^i, M^j\rangle \ll$

[7] We can choose for example $W = \sum_{i=1}^d \langle M^i\rangle$.

W with previsible density σ given by

$$\sigma_t^{ij} := \frac{d\langle M^i, M^j\rangle_t}{dW_t} \quad \text{for } i,j = 1,\dots,d, \ (0 \le t \le T).$$

Hence, σ is a symmetric, nonegative definite $d \times d$ matrix and we get

$$\langle M^i, M^j\rangle_t = \int_0^t \sigma_s^{ij}\, dW_s \quad \mathbb{P}\text{-a.s. for } i = 1,\dots,d, \ (0 \le t \le T).$$

By construction we have that $A^i \ll W$ with previsible density $\gamma^i := \alpha^i \sigma^{ii}$ for $i = 1,\dots,d$ and we get

$$A_t^i = \int_0^t \gamma_s^i\, dW_s \quad \mathbb{P}\text{-a.s. for } i = 1,\dots,d, \ (0 \le t \le T).$$

Definition 2.9 (structure condition, mean-variance tradeoff process). *Suppose X satisfies the decomposition (2.2.2) and we have (2.2.11). Then X satisfies the* ***structure condition****, if there exists a d-dimensional previsible process $\lambda = (\lambda_t)_{0\le t\le T}$, such that*

$$\sigma_t \lambda_t = \gamma_t \quad \mathbb{P}\text{-}a.s., \ (0 \le t \le T)$$

and

$$K_t := \int_0^t \lambda_s^{tr} \gamma_s\, dW_s < \infty \quad \mathbb{P}\text{-}a.s., \ (0 \le t \le T).$$

The càdlàg version of K is called the ***mean-variance tradeoff process****.*

Note, that we have

$$\begin{aligned} A_t^i &= \int_0^t \alpha_s^i\, d\langle M^i\rangle_s = \int_0^t \alpha_s^i \sigma_s^{ii}\, dW_s = \int_0^t \gamma_s^i\, dW_s \\ &= \sum_{j=1}^d \int_0^t \sigma_s^{ij} \lambda_s^j\, dW_s = \sum_{j=1}^d \int_0^t \lambda_s^j\, d\langle M^i, M^j\rangle_s \\ &=: \left(\int_0^t d\langle M\rangle_s\, \lambda_s \right)^i \quad \text{for } i = 1,\dots,d, \ (0 \le t \le T) \end{aligned}$$

and

$$\begin{aligned} K_t &= \sum_{i=1}^d \int_0^t \lambda_s^i \gamma_s^i\, dW_s = \sum_{i,j=1}^d \int_0^t \lambda_s^i \sigma_s^{ij} \lambda_s^j\, dW_s \\ &= \sum_{i,j=1}^d \int_0^t \lambda_s^i \lambda_s^j\, d\langle M^i, M^j\rangle_s \\ &=: \int_0^t \lambda_s^{tr}\, d\langle M\rangle_s\, \lambda_s = \int_0^t \lambda_s^{tr}\, dA_s \quad (0 \le t \le T). \end{aligned}$$

Since we assumed to have an arbitrage free market, there exists an equivalent martingale measure $\mathbb{Q}$ for X. To apply Girsanov's Theorem we need that $A \ll \langle M \rangle$ with a locally square integrable density process α. Hence, the structure condition is quite natural in our setting. We will later see that the structure condition is fulfilled for every continuous, adapted process X for which an equivalent local martingale measure exists.

Remark. From [Schweizer, 1994] we have the following:

(i) Assume that X satisfies the structure condition. Then K is locally bounded and does not depend on the choice of λ.

(ii) Since $\gamma_t^i := \alpha_t^i \sigma_t^{ii}$, in case of $d = 1$ we have $\sigma_t \lambda_t = \sigma_t \alpha_t$. Thus,

$$\lambda_t = \alpha_t = \frac{dA_t}{dM_t}$$

and the time discrete analogon yields

$$\frac{\Delta A_t}{\Delta M_t} = \frac{\mathbb{E}(\Delta X_t \,|\, \mathcal{F}_{t-1})}{\mathbb{V}\mathrm{ar}(\Delta X_t \,|\, \mathcal{F}_{t-1})}.$$

This gives the heuristically motivation for the name 'mean-variance tradeoff'.

(iii) K_T reflects the deviation from which X is a martingale. In particular, X satisfying the structure condition is a martingale, if and only if $K_T = 0$ $\mathbb{P}$-a.s..

Finally, we can formulate the characterization theorem for locally risk-minimizing strategies.

Theorem 2.11. *Suppose $d = 1$, X satisfies the structure condition, $\langle M \rangle$ is $\mathbb{P}$-a.s. strictly increasing, A is $\mathbb{P}$-a.s. continuous and $\mathbb{E}(K_T) < \infty$. Let ϕ be an H-admissible L^2-strategy. Then ϕ is locally risk-minimizing, if and only if ϕ is mean self-financing and the martingale $C(\phi)$ is strongly orthogonal to M.*

Proof. By assumption it holds

$$\mathbb{E}(K_T) = \mathbb{E}\left(\int_0^T |\lambda_s|^2 \, d\langle M \rangle_s \right) < \infty.$$

Hence, $\lambda = \alpha \in L^2(\mathbb{P} \otimes \langle M \rangle)$ and consequently $|\alpha| \log^+ |\alpha|$ is $(\mathbb{P} \otimes \langle M \rangle)$-integrable. Note, that from (2.2.10) $C(\phi)$ is strongly orthogonal to M, if and only if

$$\mu^C = 0 \quad (\mathbb{P} \otimes \langle M \rangle)\text{-a.e..} \tag{2.2.12}$$

Using Lemma 2.9 it remains to prove the equivalence between (2.2.8) and (2.2.12). But from Lemma 2.10 we get that the limit in (2.2.8) exists and equals $\delta^2 - 2\delta\mu^C$ $(\mathbb{P} \otimes \langle M \rangle)$-a.e.. It is left to show that (2.2.8) implies (2.2.12). We use a proof by contradiction argument, which can be seen immediately if we set $\delta := \epsilon \text{ sign } \mu^C \, \mathbb{I}_{|A| \leq k}$ and send ϵ to zero and k to infinity. $\square$

Remark. The question arises in which sense a locally risk-minimizing strategy is optimal. If ϕ is locally risk-minimizing, we have from Lemma 2.10 and Theorem 2.11 that

$$\lim_{n\to\infty} r^{\pi_n}(\xi,\delta) = \delta^2 \quad (\mathbb{P}\otimes\langle M\rangle)\text{-a.e..}$$

Thus, $\forall\, n \geq n_0(\omega,t)$, $t_i \in \pi_n$, $t \in (t_i, t_{i+1}]$ it holds for $(\mathbb{P}\otimes\langle M\rangle)$-almost all (ω,t)

$$R_{t_i}(\xi + \delta|_{(t_i,t_{i+1}]})(\omega) \geq R_{t_i}(\xi)(\omega),$$

which means that any locally pertubation of ξ leads to an increase of risk. If $\xi' \in \Xi$ is another mean self-financing strategy, such that $\delta := \xi' - \xi$ satisfies the same condition as a small pertubation, then we even have the more intuitive formulation

$$R_{t_i}(\xi + (\xi'-\xi)|_{(t_i,t_{i+1}]})(\omega) \geq R_{t_i}(\xi)(\omega),$$

which holds for $(\mathbb{P}\otimes\langle M\rangle)$-almost all (ω,t) and $\forall\, n \geq n_0(\omega,t)$, $t_i \in \pi_n$ and $t \in (t_i, t_{i+1}]$.

The strength of Theorem 2.11 is that it reduces the problem of finding a locally risk-minimizing L^2-strategy to solving the **optimality equation** (2.2.12). The Galtchouk-Kunita-Watanabe decompositions of H and $\int_0^T \xi_s\, dA_s$ with respect to $\mathbb{P}$ and M are

$$H = \mathbb{E}(H) + \int_0^T \mu_s^H\, dM_s + L_T^H \quad \mathbb{P}\text{-a.s.},$$

$$\int_0^T \xi_s\, dA_s = \mathbb{E}\left(\int_0^T \xi_s\, dA_s\right) + \int_0^T \mu_s^{\xi,A}\, dM_s + L_T^{\xi,A} \quad \mathbb{P}\text{-a.s..}$$

Hence, with

$$C_T(\phi) = H - \int_0^T \xi_s\, dX_s \quad \mathbb{P}\text{-a.s.}$$

we conclude

$$\begin{aligned}
C_T(\phi) &= \mathbb{E}(H) + \int_0^T \mu_s^H\, dM_s + L_T^H - \int_0^T \xi_s\, dM_s - \int_0^T \xi_s\, dA_s \\
&= \mathbb{E}(H) + \int_0^T \mu_s^H - \xi_s\, dM_s + L_T^H - \mathbb{E}\left(\int_0^T \xi_s\, dA_s\right) - \int_0^T \mu_s^{\xi,A}\, dM_s - L_T^{\xi,A} \\
&= C_0(\phi) + \int_0^T \mu_s^H - \xi_s - \mu_s^{\xi,A}\, dM_s + L_T^H - L_T^{\xi,A} \quad \mathbb{P}\text{-a.s..}
\end{aligned}$$

Under the assumptions of Theorem 2.11 is ϕ locally risk-minimizing, if and only if ξ solves the optimality equation

$$\mu^H - \xi - \mu^{\xi,A} = 0 \quad (\mathbb{P}\otimes\langle M\rangle)\text{-a.e..}$$

Of course, this is equivalent to the optimality equation (2.2.12). Existence and uniqueness results can be found in [Schweizer, 1991]. Here we will follow a different approach, which will lead to the same solution but in a slightly more intuitive way. Let us now return to the general case $d > 1$.

2.2.2 Pseudo-optimality and the Föllmer-Schweizer decomposition

Definition 2.10 (pseudo-optimal). *Let $H \in L^2(\mathcal{F}_T, \mathbb{P})$ be a contingent claim. Then an H-admissible L^2-strategy ϕ is called **pseudo-optimal** for H, if ϕ is mean self-financing and the martingale $C(\phi)$ is strongly orthogonal to M.*

For suitable X and $d = 1$ we have just seen in Theorem 2.11, that a locally risk-minimizing strategy coincides with the pseudo-optimal strategy. In general, pseudo-optimal strategies are easier to find as the next theorem shows.

Theorem 2.12. *An H-admissible, L^2-strategy ϕ^H is pseudo-optimal for $H \in L^2(\mathcal{F}_T, \mathbb{P})$, if and only if H admits the **Föllmer-Schweizer decomposition***

$$H = \mathbb{E}(H) + \int_0^T \xi_s^H \cdot dX_s + L_T^H \quad \mathbb{P}\text{-}a.s., \tag{2.2.13}$$

where $\xi^H \in \Xi$ and $L^H \in \mathcal{M}^2(\mathbb{P})$, with $\mathbb{E}(L_0^H) = 0$, is strongly orthogonal to $\mathcal{I}^2(M)$ with respect to $\mathbb{P}$. It is then given by

$$\phi_t^H = \big(V_t(\phi^H) - \xi_t^H \cdot X_t,\, \xi_t^H\big) \quad (0 \le t \le T),$$

where

$$V_t(\phi^H) = \mathbb{E}(H) + \int_0^t \xi_s^H \cdot dX_s + L_t^H \quad (0 \le t \le T) \tag{2.2.14}$$

and for its cost process we have $C_t(\phi^H) = \mathbb{E}(H) + L_t^H$ for $0 \le t \le T$. The remaining risk is given by

$$R_t(\phi^H) = \mathbb{E}\left(\big(L_T^H - L_t^H\big)^2 \,\Big|\, \mathcal{F}_t\right) = \mathbb{V}ar\big(L_T^H \,\big|\, \mathcal{F}_t\big) \quad (0 \le t \le T).$$

Proof. Observe

$$H = V_T(\phi^H) = C_T(\phi^H) + \int_0^T \xi_s^H \cdot dX_s = \mathbb{E}(H) + \int_0^T \xi_s^H \cdot dX_s + C_T(\phi^H) - \mathbb{E}(H).$$

Using the definition of pseudo-optimality we immediately get the assumption. □

A sufficient condition for the existence of the Föllmer-Schweizer decomposition of H is that the mean-variance tradeoff process K is uniformly bounded in t and ω. To see this we will follow [Monat and Stricker, 1995]. From now on, we always assume that K is uniformly bounded.

Note that $\Xi = L^2(M) \cap L^2(A)$, where a previsible $\mathbb{R}^d$-valued process $\xi = (\xi)_{0 \le t \le T}$ belongs to $L^2(M)$, if

$$\left(\int_0^t \xi_s^{tr} \sigma_s \xi_s \, dW_s\right)_{0 \le t \le T} \quad \text{is integrable}$$

and belongs to $L^2(A)$, if

$$\left(\int_0^t |\xi_s^{tr}\gamma_s|\, dW_s\right)_{0\le t\le T} \qquad \text{is square-integrable.}$$

Lemma 2.13. *Assume X satisfies the structure condition and that K is uniformly bounded, then $\Xi = L^2(M)$.*

Proof. For $\xi \in L^2(M)$, Cauchy-Schwarz' Inequality yields

$$\begin{aligned}\int_0^T |\xi_s^{tr}\gamma_s|\, dW_s &= \int_0^T |\xi_s^{tr}\sigma_s\lambda_s|\, dW_s\\ &\le \int_0^T \left(\xi_s^{tr}\sigma_s\xi_s\right)^{\frac12}\left(\lambda_s^{tr}\sigma_s\lambda_s\right)^{\frac12}\, dW_s\\ &\le (K_T)^{\frac12}\left(\int_0^T \xi_s^{tr}\sigma_s\xi_s\, dW_s\right)^{\frac12}.\end{aligned}$$

Thus, $L^2(M) \subset L^2(A)$ and consequently $\Xi = L^2(M)$. □

Definition 2.11 (Ψ_H)**.** *Let $0 \le T_1 \le T_2 \le T$ be previsible stopping times and assume that $H \in L^2(\mathcal{F}_{T_2-}, \mathbb{P})$. For $\xi \in L^2(M)$, look at the Galtchouk-Kunita-Watanabe decomposition of $H - \int_0^T \mathbb{I}_{(T_1,T_2)}(s)\xi_s^{tr}\, dA_s$ with respect to $\mathbb{P}$ and M*

$$H - \int_0^T \mathbb{I}_{(T_1,T_2)}(s)\xi_s^{tr}\, dA_s = \mathbb{E}\left(H - \int_0^T \mathbb{I}_{(T_1,T_2)}(s)\xi_s^{tr}\, dA_s\right) + \int_0^T \nu_s \cdot dM_s + \hat{L}_T \ \mathbb{P}\text{-a.s.},$$

where $\nu \in \Xi$, $\hat{L} \in \mathcal{M}^2(\mathbb{P})$, with $\mathbb{E}(\hat{L}_0) = 0$, is strongly orthogonal to $\mathcal{I}^2(M)$. Then we define the mapping Ψ_H as

$$\begin{aligned}\Psi_H \colon L^2(M) &\to L^2(M)\\ \Psi_H(\xi) &= \hat{\xi} := \mathbb{I}_{(T_1,T_2)}\nu.\end{aligned}$$

Since $H - \int_0^T \mathbb{I}_{(T_1,T_2)}(s)\xi_s^{tr}\, dA_s$ is $\mathcal{F}_{T_2-}$-measurable, its Galtchouk-Kunita-Watanabe decomposition can be rewritten as

$$\begin{aligned}&H - \int_0^T \mathbb{I}_{(T_1,T_2)}(s)\xi_s^{tr}\, dA_s\\ &\quad = \mathbb{E}\left(H - \int_0^T \mathbb{I}_{(T_1,T_2)}(s)\xi_s^{tr}\, dA_s\right) + \int_0^T \mathbb{I}_{(0,T_2)}(s)\nu_s \cdot dM_s + \hat{L}_0 + \int_0^T \mathbb{I}_{(0,T_2)}(s)\, d\hat{L}_s.\end{aligned}$$

If we define $\hat{H}$ by

$$\hat{H} := \mathbb{E}\left(H - \int_0^T \mathbb{I}_{(T_1,T_2)}(s)\xi_s^{tr}\, dA_s\right) + \hat{L}_0 + \int_0^T \mathbb{I}_{(0,T_1]}(s)\nu_s \cdot dM_s + \int_0^T \mathbb{I}_{(0,T_1]}(s)\, d\hat{L}_s,$$

then $\hat{H}$ is $\mathcal{F}_{T_1}$-measurable and

$$H - \int_0^T \mathbb{I}_{(T_1,T_2)}(s)\xi_s^{tr}\, dA_s = \hat{H} + \int_0^T \mathbb{I}_{(T_1,T_2)}(s)\hat{\xi}_s \cdot dM_s + \int_0^T \mathbb{I}_{(T_1,T_2)}(s)\, d\hat{L}_s \quad \mathbb{P}\text{-a.s.}.$$

Note, that we switched to the other formulation of the Föllmer-Schweizer decomposition (compare Equation (2.1.4)). Now suppose $\xi \in L^2(M)$ is a fixed point of Ψ_H. Then the last equation yields

$$H = \hat{H} + \int_0^T \mathbb{I}_{(T_1,T_2)}(s)\xi_s \cdot dX_s + \int_0^T \mathbb{I}_{(T_1,T_2)}(s)\, d\hat{L}_s \quad \mathbb{P}\text{-a.s.}. \tag{2.2.15}$$

Conversely, if Equation (2.2.15) holds for $\xi \in L^2(M)$, the Galtchouk-Kunita-Watanabe decomposition of

$$\hat{H} = \mathbb{E}(\hat{H}) + \int_0^T \mathbb{I}_{(0,T_1]}(s)\theta_s \cdot dM_s + N_{T_1} \quad \mathbb{P}\text{-a.s.}$$

yields

$$\begin{aligned} &H - \int_0^T \mathbb{I}_{(T_1,T_2)}(s)\xi_s^{tr}\, dA_s \\ &= \mathbb{E}(\hat{H}) + \int_0^T \left(\mathbb{I}_{(0,T_1]}(s)\theta_s + \mathbb{I}_{(T_1,T_2)}(s)\xi_s\right) \cdot dM_s + \left(N_{T_1} + \int_0^T \mathbb{I}_{(T_1,T_2)}(s)\, d\hat{L}_s\right) \quad \mathbb{P}\text{-a.s.}. \end{aligned}$$

Since the Galtchouk-Kunita-Watanabe decomposition is unique, we obtain $\nu = \mathbb{I}_{(0,T_1]}\theta + \mathbb{I}_{(T_1,T_2)}\xi$ and thus it holds $\Psi_H(\xi) = \xi$.

To prove the existence and uniqueness of the Föllmer-Schweizer decomposition we need some auxiliary results, which are dealt with in the next two lemmata.

Lemma 2.14. *Assume X satisfies the structure condition. Let $0 \le T_1 \le T_2 \le T$ be previsible stopping times and let $H \in L^2(\mathcal{F}_{T_2-}, \mathbb{P})$. If there $\exists\, b \in (0,1)$, such that $K_{T_2-} - K_{T_1} \le b$ $\mathbb{P}$-a.s., then Ψ_H has a unique fixed point.*

Proof. Define the norm

$$\|\xi\|_{L^2(M)} := \left\| \int_0^T \xi_s \cdot dM_s \right\|_2 .$$

Then $\left(L^2(M), \|\cdot\|_{L^2(M)}\right)$ is a Banach space. With Banach's Fixed Point Theorem it remains to prove that Ψ_H is a contraction on $L^2(M)$. Take $\xi, \xi' \in L^2(M)$ and recall

$\Psi_H(\xi) = \hat{\xi}$, $\Psi_H(\xi') = \hat{\xi}'$, then we have

$$
\begin{aligned}
&\|\hat{\xi} - \hat{\xi}'\|^2_{L^2(M)} \\
&\quad = \left\| \int_0^T \left(\hat{\xi}_s - \hat{\xi}'_s\right) \cdot dM_s \right\|_2^2 \\
&\quad \le \left\| \int_0^T \left(\hat{\xi}_s - \hat{\xi}'_s\right) \cdot dM_s \right\|_2^2 + \|\hat{H} - \hat{H}'\|_2^2 + \left\| \int_0^T \mathbb{I}_{(T_1,T_2)}(s)\, d\hat{L}_s - \int_0^T \mathbb{I}_{(T_1,T_2)}(s)\, d\hat{L}'_s \right\|_2^2 \\
&\quad = \left\| \int_0^T \left(\hat{\xi}_s - \hat{\xi}'_s\right) \cdot dM_s + (\hat{H} - \hat{H}') + \left(\int_0^T \mathbb{I}_{(T_1,T_2)}(s)\, d\hat{L}_s - \int_0^T \mathbb{I}_{(T_1,T_2)}(s)\, d\hat{L}'_s \right) \right\|_2^2 \\
&\quad = \left\| \int_0^T \mathbb{I}_{(T_1,T_2)}(s)(\xi_s - \xi'_s)^{tr}\, dA_s \right\|_2^2 \\
&\quad = \left\| \int_0^T \mathbb{I}_{(T_1,T_2)}(s)(\xi_s - \xi'_s)^{tr} \sigma_s \lambda_s\, dW_s \right\|_2^2 \\
&\quad \le \mathbb{E}\left(\int_0^T \mathbb{I}_{(T_1,T_2)}(s) \lambda_s^{tr} \sigma_s \lambda_s\, dW_s \int_0^T (\xi_s - \xi'_s)^{tr} \sigma_s (\xi_s - \xi'_s)\, dW_s \right) \\
&\quad \le \|K_{T_2-} - K_{T_1}\|_\infty \mathbb{E}\left(\int_0^T (\xi_s - \xi'_s)^{tr} \sigma_s (\xi_s - \xi'_s)\, dW_s \right) \\
&\quad \le b \|\xi - \xi'\|^2_{L^2(M)},
\end{aligned}
$$

where we used orthogonality, the definition of the structure condition and the symmetry of the nonnegative matrix σ. □

Lemma 2.15. *Assume X satisfies the structure condition and K is uniformly bounded. Let $0 \le T_0 \le T$ be a previsible stopping time and let $H \in L^2(\mathcal{F}_{T_0}, \mathbb{P})$. Then*

$$H = \tilde{H} + \int_0^T \mathbb{I}_{T_0}(s) \tilde{\xi}_s \cdot dX_s + \tilde{L}_{T_0} \quad \mathbb{P}\text{-a.s.}, \tag{2.2.16}$$

where $\tilde{H} \in L^2(\mathcal{F}_{T_0-}, \mathbb{P})$, $\tilde{\xi} \in \Xi$ and $\tilde{L} \in \mathcal{M}^2(\mathbb{P})$, which is equal to zero[8] *on the interval $[0, T_0)$ and is strongly orthogonal to $\mathcal{I}^2(M)$. Further, the decomposition is unique in the sense that if also*

$$H = \tilde{H}' + \int_0^T \mathbb{I}_{T_0}(s) \tilde{\xi}'_s \cdot dX_s + \tilde{L}'_{T_0} \quad \mathbb{P}\text{-a.s.},$$

where $(\tilde{H}', \tilde{\xi}', \tilde{L}')$ satisfies the same conditions as $(\tilde{H}, \tilde{\xi}, \tilde{L})$, then it holds

$$
\begin{aligned}
\tilde{H} &= \tilde{H}' \quad \mathbb{P}\text{-a.s.}, \\
\mathbb{I}_{T_0} \tilde{\xi} &= \mathbb{I}_{T_0} \tilde{\xi}' \quad \text{in } L^2(M), \\
\tilde{L}_{T_0} &= \tilde{L}'_{T_0} \quad \mathbb{P}\text{-a.s.}.
\end{aligned}
$$

[8] Compare Equation (2.1.4).

Proof. Since $(\mathbb{E}(H \,|\, \mathcal{F}_t))_{0 \le t \le T}$ is a square-integrable martingale we have

$$H = \mathbb{E}(H \,|\, \mathcal{F}_T) = \mathbb{E}(H) + \int_0^T \tilde{\xi}_s \cdot dM_s + L_T \quad \mathbb{P}\text{-a.s.},$$

where $\tilde{\xi} \in L^2(M)$ and $L \in \mathcal{M}^2(\mathbb{P})$, with $\mathbb{E}(L_0) = 0$, is strongly orthogonal to $\mathcal{I}^2(M)$. By assumption we have $H \in L^2(\mathcal{F}_{T_0}, \mathbb{P})$. Thus, $\tilde{\xi}$ equals zero on the interval $(T_0, T]$ and $L_{T_0} = L_T$ $\mathbb{P}$-a.s. and we get

$$H = \mathbb{E}(H) + \int_0^T \mathbb{I}_{(0,T_0]}(s)\tilde{\xi}_s \cdot dM_s + L_{T_0} \quad \mathbb{P}\text{-a.s.}.$$

Further,

$$\mathbb{E}(H \,|\, \mathcal{F}_{T_0-}) = \mathbb{E}(H) + \int_0^T \mathbb{I}_{(0,T_0)}(s)\tilde{\xi}_s \cdot dM_s + L_{T_0-} \quad \mathbb{P}\text{-a.s.}.$$

Rearranging the last equation by $\mathbb{E}(H)$ and inserting it into the decomposition of H yields

$$H = \mathbb{E}(H \,|\, \mathcal{F}_{T_0-}) + \int_0^T \mathbb{I}_{T_0}(s)\tilde{\xi}_s \cdot dM_s + L_{T_0} - L_{T_0-} \quad \mathbb{P}\text{-a.s.}.$$

From Lemma 2.13 we know $L^2(M) = \Xi$ and since A is previsible, $\int_0^T \mathbb{I}_{T_0}(s)\tilde{\xi}_s^{tr}\, dA_s$ is $\mathcal{F}_{T_0-}$-measurable. Thus, we get the desired decomposition

$$H = \underbrace{\mathbb{E}(H \,|\, \mathcal{F}_{T_0-}) - \int_0^T \mathbb{I}_{T_0}(s)\tilde{\xi}_s^{tr}\, dA_s}_{=: \tilde{H}} + \int_0^T \mathbb{I}_{T_0}(s)\tilde{\xi}_s \cdot dX_s + \underbrace{L_{T_0} - L_{T_0-}}_{=: \tilde{L}_{T_0}} \quad \mathbb{P}\text{-a.s.}.$$

Now let us prove the uniqueness. Using substraction, we can assume without loss of generality $H = 0$ $\mathbb{P}$-a.s.. Then

$$0 = \tilde{H} + \int_0^T \mathbb{I}_{T_0}(s)\tilde{\xi}_s \cdot dX_s + \tilde{L}_{T_0} \quad \mathbb{P}\text{-a.s.},$$

and hence

$$0 = \mathbb{E}(0 \,|\, \mathcal{F}_{T_0-}) = \tilde{H} + \int_0^T \mathbb{I}_{T_0}(s)\tilde{\xi}_s^{tr}\, dA_s \quad \mathbb{P}\text{-a.s.}.$$

Substracting the last equation from the previous one yields

$$0 = \int_0^T \mathbb{I}_{T_0}(s)\tilde{\xi}_s \cdot dM_s + \tilde{L}_{T_0} \quad \mathbb{P}\text{-a.s.}$$

and since the Galtchouk-Kunita-Watanabe decomposition is unique we get

$$\begin{aligned}\int_0^T \mathbb{I}_{T_0}(s)\tilde{\xi}_s \cdot dM_s &= 0 \quad \mathbb{P}\text{-a.s.},\\ \tilde{L}_{T_0} &= 0 \quad \mathbb{P}\text{-a.s..}\end{aligned}$$

The structure condition finally yields

$$\begin{aligned} 0 &\leq \left| \int_0^T \mathbb{I}_{T_0}(s)\tilde{\xi}_s^{tr}\, dA_s \right| \\ &= \left| \int_0^T \mathbb{I}_{T_0}(s)\tilde{\xi}_s^{tr}\gamma_s\, dW_s \right| \\ &= \left| \int_0^T \mathbb{I}_{T_0}(s)\tilde{\xi}_s^{tr}\sigma_s\lambda_s\, dW_s \right| \\ &\leq \left(\int_0^T \mathbb{I}_{T_0}(s)\tilde{\xi}_s^{tr}\sigma_s\tilde{\xi}_s\, dW_s \right)^{\frac{1}{2}} \left(\int_0^T \lambda_s^{tr}\sigma_s\lambda_s\, dW_s \right)^{\frac{1}{2}} \\ &\leq \left(\left\langle \int_0^{\cdot} \mathbb{I}_{T_0}(s)\tilde{\xi}_s \cdot dM_s \right\rangle_T \right)^{\frac{1}{2}} (K_T)^{\frac{1}{2}} \\ &= 0, \end{aligned}$$

where we used that K_T is bounded and $\left\langle \int_0^{\cdot} \mathbb{I}_{T_0}(s)\tilde{\xi}_s \cdot dM_s \right\rangle_T = 0$. Thus, we also have $\tilde{H} = 0$ $\mathbb{P}$-a.s. and hence the decomposition is unique. □

We can now formulate the corresponding theorem. The idea is to use a chopping technique: The result is easy if $\|K_T\|_\infty < 1$. Since K is uniformly bounded there are only finitely many jumps and we differ between jumps less than a constant $b \in (0,1)$ and jumps exceeding b. Hence, we divide the interval $[0,T]$ into subintervals by a suitable sequence of stopping times, such that on each subinterval the growth of K is bounded by $b < 1$ and on the boundaries we deal with the jumps exceeding b. Finally, we use a backward induction argument.

Theorem 2.16. *Assume X satisfies the structure condition and K is uniformly bounded in t and ω. Then, every $H \in L^2(\mathcal{F}_T, \mathbb{P})$ admits a Föllmer-Schweizer decomposition and it is unique in the sense that if*

$$H = \mathbb{E}(H) + \int_0^T \xi_s^H \cdot dX_s + L_T^H = \mathbb{E}(H) + \int_0^T \xi_s^{H'} \cdot dX_s + L_T^{H'} \quad \mathbb{P}\text{-}a.s.,$$

where $\left(\xi^{H'}, L^{H'}\right)$ satisfies the same conditions as $\left(\xi^H, L^H\right)$, then it holds

$$\begin{aligned} L_0^H &= L_0^{H'} \quad \mathbb{P}\text{-}a.s.,\\ \xi^H &= \xi^{H'} \quad in\ L^2(M),\\ L_T^H &= L_T^{H'} \quad \mathbb{P}\text{-}a.s.. \end{aligned}$$

Proof. Observe, since K is uniformly bounded there $\exists\, b \in (0,1)$ and a finite sequence $(T_n)_{0\le n\le N}$ of previsible stopping times, such that for $0 = T_0 \le T_1 \le \cdots \le T_N = T$

$$K_{T_n-} - K_{T_{n-1}} \le b \quad \mathbb{P}\text{-a.s., } n = 1,\ldots,N.$$

Now let $\beta \in (0,1)$ and construct the sequence

$$\begin{aligned} T_0 &= 0 \\ T_{n+1} :&= \begin{cases} \inf\{T_n < t \le T \,|\, K_t - K_{T_n} \ge \beta\} \\ T, \text{ if this set is empty} \end{cases} \\ T_N &= T, \end{aligned}$$

where N is large enough, such that $K_{T-} - K_{T_{N-1}} \le \beta$ $\mathbb{P}$-a.s.. By definition is each stopping time T_n previsible.
Since $H \in L^2(\mathcal{F}_{T_N}, \mathbb{P})$, Lemma 2.15 yields

$$H = \tilde{H}^N + \int_0^T \mathbb{I}_{T_N}(s)\tilde{\xi}_s^N \cdot dX_s + \tilde{L}_{T_N}^N \quad \mathbb{P}\text{-a.s..}$$

Now using Lemma 2.14 and Equation (2.2.15) between T_N and T_{N-1} we can rewrite $\tilde{H}^N$ as

$$\tilde{H}^N = \hat{H}^{N-1} + \int_0^T \mathbb{I}_{(T_{N-1},T_N)}(s)\hat{\xi}_s^{N-1} \cdot dX_s + \int_0^T \mathbb{I}_{(T_{N-1},T_N)}(s)\, d\hat{L}_s^{N-1}$$

and we get

$$\begin{aligned} H = \hat{H}^{N-1} &+ \int_0^T \left(\mathbb{I}_{(T_{N-1},T_N)}(s)\hat{\xi}_s^{N-1} + \mathbb{I}_{T_N}(s)\tilde{\xi}_s^N\right) \cdot dX_s \\ &+ \left(\tilde{L}_{T_N}^N + \int_0^T \mathbb{I}_{(T_{N-1},T_N)}(s)\, d\hat{L}_s^{N-1}\right). \end{aligned}$$

By recursively using Lemma 2.15 and 2.14 we obtain

$$\begin{aligned} H = &\underbrace{\hat{H}^0}_{=\mathbb{E}(H)+\hat{L}_0} + \int_0^T \underbrace{\sum_{n=1}^N \left(\mathbb{I}_{(T_{n-1},T_n)}(s)\hat{\xi}_s^{n-1} + \mathbb{I}_{T_n}(s)\tilde{\xi}_s^n\right)}_{=:\,\xi_s^H} \cdot dX_s \\ &+ \sum_{n=1}^N \left(\tilde{L}_{T_n}^n + \int_0^T \mathbb{I}_{(T_{n-1},T_n)}(s)\, d\hat{L}_s^{n-1}\right) \\ = &\ \mathbb{E}(H) + \int_0^T \xi_s^H \cdot dX_s + \underbrace{\hat{L}_0 + \sum_{n=1}^N \left(\tilde{L}_{T_n}^n + \int_0^T \mathbb{I}_{(T_{n-1},T_n)}(s)\, d\hat{L}_s^{n-1}\right)}_{=:\,L_T^H}, \end{aligned}$$

where $\xi^H \in \Xi$ and $L^H \in \mathcal{M}^2(\mathbb{P})$, with $\mathbb{E}(L_0^H) = \mathbb{E}(\mathbb{E}(L_T^H \mid \mathcal{F}_0)) = \mathbb{E}(\hat{L}_0) = 0$, is strongly orthogonal to $\mathcal{I}^2(M)$.

Now let us prove the uniqueness of the Föllmer-Schweizer decomposition. Using substraction, we can assume without loss of generality $H = 0$ $\mathbb{P}$-a.s. and we have

$$\mathbb{E}(H) + L_0^H + \int_0^T \xi_s^H \cdot dX_s + \underbrace{L_T^H - L_0^H}_{\bar{L}_T^H} = 0 \quad \mathbb{P}\text{-a.s..}$$

Rewriting yields

$$\begin{aligned}\Bigg(\mathbb{E}(H) + L_0^H + \int_0^T \mathbb{I}_{(0,T_N)}(s)\xi_s^H \cdot dX_s + \bar{L}_{T_N-}^H\Bigg)& \\ + \int_0^T \mathbb{I}_{T_N}(s)\xi_s^H \cdot dX_s + \big(\bar{L}_T^H - \bar{L}_{T_N-}^H\big) &= 0 \quad \mathbb{P}\text{-a.s..}\end{aligned}$$

Since the first summand is in $L^2(\mathcal{F}_{T_N-}, \mathbb{P})$ we obtain as in the proof of Lemma 2.15

$$\begin{aligned}\mathbb{E}(H) + L_0^H + \int_0^T \mathbb{I}_{(0,T_N)}(s)\xi_s^H \cdot dX_s + \bar{L}_{T_N-}^H &= 0 \quad \mathbb{P}\text{-a.s.,} \\ \int_0^T \mathbb{I}_{T_N}(s)\xi_s^H \cdot dM_s &= 0 \quad \mathbb{P}\text{-a.s.,} \\ \bar{L}_T^H &= \bar{L}_{T_N-}^H \quad \mathbb{P}\text{-a.s..}\end{aligned}$$

Consequently,

$$\begin{aligned}&\Bigg(\mathbb{E}(H) + L_0^H + \int_0^T \mathbb{I}_{(0,T_{N-1}]}(s)\xi_s^H \cdot dX_s + \bar{L}_{T_{N-1}}^H\Bigg) + \int_0^T \mathbb{I}_{(T_{N-1},T_N)}(s)\xi_s^H \cdot dX_s \\ &+ \int_0^T \mathbb{I}_{(T_{N-1},T_N)}(s)\, d\bar{L}_s^H = 0 \quad \mathbb{P}\text{-a.s.}\end{aligned}$$

and since the decomposition (2.2.15) is unique we obtain

$$\begin{aligned}\mathbb{E}(H) + L_0^H + \int_0^T \mathbb{I}_{(0,T_{N-1}]}(s)\xi_s^H \cdot dX_s + \bar{L}_{T_{N-1}}^H &= 0 \quad \mathbb{P}\text{-a.s.,} \\ \int_0^T \mathbb{I}_{(T_{N-1},T_N]}(s)\xi_s^H \cdot dM_s &= 0 \quad \mathbb{P}\text{-a.s.,} \\ \bar{L}_T^H &= \bar{L}_{T_{N-1}}^H \quad \mathbb{P}\text{-a.s..}\end{aligned}$$

Applying induction this proves

$$\begin{aligned}\underbrace{\mathbb{E}(H)}_{=0} + L_0^H &= 0 \quad \mathbb{P}\text{-a.s.,} \\ \int_0^T \mathbb{I}_{(T_0,T_N]}(s)\xi_s^H \cdot dM_s &= 0 \quad \mathbb{P}\text{-a.s.,} \\ L_T^H = \bar{L}_T^H = \bar{L}_{T_0}^H = \bar{L}_0^H &= 0 \quad \mathbb{P}\text{-a.s.}\end{aligned}$$

and thus the Föllmer-Schweizer-decomposition is unique. □

Remark. In case of K beeing not uniformly bounded, the Föllmer-Schweizer decomposition may not exist. To see this, assume $d = 1$ and suppose we are at a time u, where K jumps. Further, suppose $H = \xi_u \Delta M_u$ with $\xi \in L^2(M)$. The Föllmer-Schweizer decomposition of H is

$$H = \mathbb{E}(H) + \int_0^T \xi_s^H \, dX_s + L_T^H \quad \mathbb{P}\text{-a.s..}$$

But since $\mathbb{E}(H \,|\, \mathcal{F}_{u-}) = 0$, we have

$$H = \xi_u \Delta M_u = \xi_u^H \Delta X_u + \Delta L_u^H.$$

Then $L^H \in \mathcal{M}^2(\mathbb{P})$, with $\mathbb{E}(L_0^H) = 0$, is strongly orthogonal to $\mathcal{I}^2(M)$, which implies $\xi_u = \xi_u^H$. However, from the structure condition in the case $d = 1$ we get $\Delta A_u = \lambda_u \Delta\langle M\rangle_u$ and $\Delta K_u = \lambda_u^2 \Delta\langle M\rangle_u$. Therefore,

$$\mathbb{E}\left(\xi_u^2 (\Delta A_u)^2\right) = \mathbb{E}\left(\xi_u^2 \Delta\langle M\rangle_u \Delta K_u\right).$$

But since $\xi_u^2 \Delta\langle M\rangle_u$ spans $L^1(\mathcal{F}_{u-})$, $\mathbb{E}\left(\xi_u^2 (\Delta A_u)^2\right)$ is only bounded, if and only if ΔK_u is uniformly bounded. Thus, H may not have a Föllmer-Schweizer decomposition.

2.2.3 The minimal martingale measure

Because of the Theorems 2.11 and 2.12 we are heavily interested in finding the Föllmer-Schweizer decomposition of a contingent claim H. We will see a constructive way by switching to a specific martingale measure. This approach works especially very well, if we assume that X is **continuous** and K is **uniformly bounded**. Let us first only assume continuity of X. The following theorem is based on [Schweizer, 1994].

Theorem 2.17. *Suppose X is a continuous semimartingale with decomposition* (2.2.2) *and $Z^* \in \mathcal{M}^2_{loc}(\mathbb{P})$ with $Z^* > 0$ and $Z_0^* = 1$ $\mathbb{P}$-a.s., such that the product $Z^* X^i$ is a local $\mathbb{P}$-martingale for $i = 1, \ldots, d$. Then X satisfies the structure condition and $\alpha^i \in L^2_{loc}(M^i)$ for $i = 1, \ldots, d$. Furthermore, Z^* has the representation*

$$Z^* = \mathcal{E}\left(-\int \lambda \cdot dM\right) \mathcal{E}(L),$$

where $L \in \mathcal{M}^2_{loc}(\mathbb{P})$, with $\mathbb{E}(L_0) = 0$, is strongly orthogonal to M^i for all $i = 1, \ldots, d$.

Proof. First, we need to find the previsible, symmetric, nonnegative definite $d \times d$ matrix density σ. Let $N^1, \ldots, N^d \in \mathcal{M}^2_{0,\mathrm{loc}}(\mathbb{P})$ be pairwise strongly orthogonal, such that each M^i is in the stable subspace of $\mathcal{M}_0^2(\mathbb{P})$ generated by $N^1, \dots, N^d$. Then each M^i has a representation

$$M^i = \sum_{j=1}^{d} \int \rho^{ij} \, dN^j,$$

where $\rho = (\rho^{ij})_{i,j=1,\dots,d}$ is a previsible process with $\rho^{ij} \in L^2_{\mathrm{loc}}(\mathbb{P})$ for each pair ij. Since X is continuous, also M is continuous. Thus, we can assume that N is continuous as well. Choose an increasing previsible càdlàg process W with $W_0 = 0$ and $\langle N^i \rangle \ll W$ for each i and define

$$\zeta_t^i := \frac{d\langle N^i \rangle_t}{dW_t} \quad \text{for } i = 1, \dots, d.$$

We can assume without loss of generality $\zeta_t^i \in \{0, 1\}$ for all i, t, by simply replacing N^i with $\int \mathbb{I}_{\{\zeta^i \neq 0\}} \frac{1}{\sqrt{\zeta^i}} \, dN^i$. Since

$$\int \mathbb{I}_{\{\zeta^j = 0\}} \, d\langle N^j \rangle = \int \zeta^j \mathbb{I}_{\{\zeta^j = 0\}} \, dW = 0, \tag{2.2.17}$$

we can further assume $\rho_t^{ij} = 0$ on the set $\{\zeta_t^j = 0\}$ for all i, j, t. Thus, it holds

$$\rho_t^{ij} \zeta_t^j = \rho_t^{ij} \quad \forall \, i, j, t$$

and hence we have

$$\langle M^i, M^j \rangle = \sum_{k=1}^{d} \int \rho^{ik} \rho^{jk} \, d\langle N^k \rangle = \sum_{k=1}^{d} \int \rho^{ik} \rho^{jk} \zeta^k \, dW = \int \left(\rho \rho^{tr} \right)^{ij} \, dW, \tag{2.2.18}$$

where we used the pairwise strong orthogonality of $N^i, N^j \; \forall \, i \neq j$. Thus, we conclude

$$\sigma_t = \rho_t \rho_t^{tr} \quad \mathbb{P}\text{-a.s.}, \; (0 \leq t \leq T). \tag{2.2.19}$$

Next, we need the previsible densities γ^i, for $i = 1, \dots, d$. Define $U := \int \frac{1}{Z^*_-} \, dZ^*$ with $U_0 = 0$. Then $U \in \mathcal{M}^2_{0,\mathrm{loc}}(\mathbb{P})$ since $Z^* \in \mathcal{M}^2_{\mathrm{loc}}(\mathbb{P})$. U is well defined, since $Z^* > 0$. Observe, $Z^* = \mathcal{E}(U)$. The Galtchouk-Kunita-Watanabe decomposition of U can be written as

$$U = -\sum_{j=1}^{d} \int \psi^j \, dN^j + R,$$

where $\psi^j \in L^2_{\mathrm{loc}}(N^j)$ and $R \in \mathcal{M}^2_{\mathrm{loc}}(\mathbb{P})$, with $\mathbb{E}(R_0) = 0$, is strongly orthogonal to $\mathcal{I}^2(N^j)$ for each j. From (2.2.17) we get that we can choose

$$\psi_t^j = 0 \quad \text{on the set } \{\zeta_t^j = 0\}, \, \forall \, j, t. \tag{2.2.20}$$

Applying the product rule to Z^* and $X^i = X_0^i + M^i + A^i$ yields

$$d(Z^* X^i) = \big(X_-^i \, dZ^* + Z_-^* \, dM^i + d[Z^*, A^i]\big) + Z_-^* \, dA^i + d[Z^*, M^i]. \tag{2.2.21}$$

The product $Z^* X^i$ is a local $\mathbb{P}$-martingale. From [Protter, 2005] we know that $[Z^*, A^i]$ is a local $\mathbb{P}$-martingale, since Z^* is a local $\mathbb{P}$-martingale and A is a previsible process of bounded variation. Thus, the bracket term on the right hand side is the differential of a local $\mathbb{P}$-martingale. By the definition of U we have

$$[Z^*, M^i] = \int Z_-^* \, d[U, M^i].$$

Further, it holds

$$\begin{aligned} [U, M^i] &= \left[-\sum_{j=1}^d \int \psi^j \, dN^j + R, \sum_{k=1}^d \int \rho^{ik} \, dN^k \right] \\ &= -\sum_{j=1}^d \int \psi^j \rho^{ij} \, d\langle N^j \rangle + \left[R, \sum_{k=1}^d \int \rho^{ik} \, dN^k \right] \\ &= -\sum_{j=1}^d \int \rho^{ij} \psi^j \, dW, \end{aligned}$$

where we used the pairwise strong orthogonality of $N^j, N^k \; \forall \, j \neq k$ and the strong orthogonality of R to $\mathcal{I}^2(N^k)$ for each k. Together with (2.2.21) and the fact that $Z_-^* > 0$ we conclude

$$A^i = \int \underbrace{(\rho\psi)^i}_{=: \gamma^i} \, dW \quad \text{for } i = 1, \ldots, d.$$

Finally, let us check the assumptions of the structure condition. Denote by $\bar{\psi}$ the projection of ψ on Ker ρ. Then $\psi = \bar{\psi} + \nu$ for some previsible process ν, with $\rho\nu = 0$. Since

$$(\text{Ker } \rho)^\perp = \text{Im } \rho^{tr},$$

there exists a previsible process λ, such that

$$\psi = \bar{\psi} + \nu = \rho^{tr} \lambda + \nu. \tag{2.2.22}$$

Thus, with (2.2.19) we obtain

$$A^i = \int (\rho\psi)^i \, dW = \int (\sigma\lambda)^i \, dW \quad \text{for } i = 1, \ldots, d.$$

Observe, that on the set $\{\sigma_t^{ii} = 0\}$ we have $\sigma_t^{ij} = 0$. This is true since

$$0 = \sigma^{ii} = \sum_{k=1}^{d} \left(\rho^{ik}\right)^2$$

implies $\rho^{ik} = 0 \; \forall \, k = 1, \ldots, d$, and hence

$$\sigma^{ij} = \sum_{k=1}^{d} \rho^{ik} \rho^{jk} = 0.$$

Thus, the density

$$\alpha^i := \frac{(\sigma\lambda)^i}{\sigma^{ii}}$$

is well defined and we conclude with the Kunita-Watanabe inequality that $A^i \ll \langle M^i \rangle$ and $\sigma_t \lambda_t = \gamma_t$ $\mathbb{P}$-a.s. for $0 \le t \le T$. Further, we obtain

$$\begin{aligned}
\int \left(\alpha^i\right)^2 \, d\langle M^i \rangle &= \int \frac{\left((\rho\psi)^i\right)^2}{(\sigma^{ii})^2} \, d\langle M^i \rangle \\
&= \int \frac{1}{\sigma^{ii}} \left((\rho\psi)^i\right)^2 \, dW \\
&\le \int \frac{1}{\sigma^{ii}} \sum_{j=1}^{d} \left(\rho^{ij}\right)^2 \sum_{j=1}^{d} \left(\psi^j\right)^2 \, dW \\
&= \sum_{j=1}^{d} \int \left(\psi^j\right)^2 \, dW \\
&= \sum_{j=1}^{d} \int \left(\psi^j\right)^2 \, d\langle N^j \rangle,
\end{aligned}$$

where we used the definition of α^i, (2.2.18), the Cauchy-Schwarz Inequality, (2.2.19) and (2.2.20). Since $\psi^j \in L^2_{\text{loc}}(N^j)$, the above inequality yields $\alpha^i \in L^2_{\text{loc}}(M^i)$ for each

i. Similarly, we obtain with (2.2.18), (2.2.19), (2.2.22) and (2.2.20) that

$$\begin{aligned} K = \sum_{i,j=1}^{d} \int \lambda^i \lambda^j \, d\langle M^i, M^j \rangle &= \sum_{i,j=1}^{d} \int \lambda^i \left(\rho \rho^{tr}\right)^{ij} \lambda^j \, dW \\ &= \int \lambda^{tr} \sigma \lambda \, dW \\ &= \int \left\| \rho^{tr} \lambda \right\|^2 \, dW \\ &\leq \int \|\psi\|^2 \, dW \\ &= \sum_{j=1}^{d} \int \left(\psi^j\right)^2 \, d\langle N^j \rangle. \end{aligned}$$

Hence, $\lambda \in L^2_{\text{loc}}(M)$, which in particular implies that the process $\int \lambda \cdot dM \in \mathcal{M}^2_{0,\text{loc}}(\mathbb{P})$ is well-defined.
What is left to show is the representation of Z^*. Let $Y \in \mathcal{M}^2_{0,\text{loc}}(\mathbb{P})$ and look at

$$\begin{aligned} \left\langle Y, \int \lambda \cdot dM \right\rangle &= \sum_{i=1}^{d} \int \lambda^i \, d\langle Y, M^i \rangle \\ &= \sum_{i,j=1}^{d} \int \lambda^i \rho^{ij} \, d\langle Y, N^j \rangle \\ &= \left\langle Y, \sum_{j=1}^{d} \int \left(\rho^{tr} \lambda\right)^j \, dN^j \right\rangle. \end{aligned}$$

By comparison of coefficients we get

$$\int \lambda \cdot dM = \sum_{j=1}^{d} \int \left(\rho^{tr} \lambda\right)^j \, dN^j.$$

Then U simplifies with (2.2.22) to

$$U = -\int \lambda \cdot dM + \underbrace{R - \sum_{j=1}^{d} \int \nu^j \, dN^j}_{=: L},$$

where we get with $\rho\nu = 0$ that L is strongly orthogonal to $\mathcal{I}^2(N^k)$ for each k and hence also to M^k. By (2.2.22) we have $\nu^j \in L^2_{\text{loc}}(N^j)$ for each j and since $Z^* \in \mathcal{M}^2_{\text{loc}}(\mathbb{P})$ we have $L \in \mathcal{M}^2_{\mathrm{loc}}(\mathbb{P})$, with $\mathbb{E}(L_0) = 0$. From [Jacod and Shiryaev, 2003, p. 53 ff.] we will use the identity for two square integrable local martingales[9] X, Y

$$[X,Y]_t = \langle X^c, Y^c \rangle_t + \sum_{s \le t} \Delta X_s \Delta Y_s. \tag{2.2.23}$$

Since M is continuous, also $\int \lambda \cdot dM$ is continuous. Hence we have

$$\left[L, \int \lambda \cdot dM\right] = \left\langle L^c, \int \lambda \cdot dM \right\rangle = \left\langle L, \int \lambda \cdot dM \right\rangle = 0,$$

since L is strongly orthogonal to each M^j. Together with the product rule formula $\mathcal{E}(X + Y + [X,Y]) = \mathcal{E}(X)\mathcal{E}(Y)$, where X, Y are two semimartingales, we get

$$Z^* = \mathcal{E}(U) = \mathcal{E}\left(-\int \lambda \cdot dM + L\right) = \mathcal{E}\left(-\int \lambda \cdot dM\right)\mathcal{E}(L).$$

□

Now we look at the stochastic differential equation

$$\begin{aligned} dZ_t &= -Z_{t-}\lambda_t \cdot dM_t, \\ Z_0 &= 1 \quad \mathbb{P}\text{-a.s..} \end{aligned} \tag{2.2.24}$$

Its unique solution is given by

$$Z_t = \mathcal{E}\left(-\int \lambda \cdot dM\right)_t \quad (0 \le t \le T).$$

Since X is continuous, we have that M is continuous and hence also $-\int \lambda \cdot dM$. As we have seen in the above proof it holds $-\int \lambda \cdot dM \in \mathcal{M}^2_{0,\mathrm{loc}}(\mathbb{P})$. Using the theorem of exponential processes for continuous semimartingales we get that Z is a local martingale with respect to $\mathbb{P}$. By continuity of M we further have $Z > 0$. Note, that by construction of Z we get that the product ZX^i is a local martingale with respect to $\mathbb{P}$ for $i = 1, \dots, d$.

Now assume that K is uniformly bounded and look at the solution of (2.2.24)

$$\begin{aligned} Z_t = \mathcal{E}\left(-\int \lambda \cdot dM\right)_t &= \exp\left(-\int_0^t \lambda_s \cdot dM_s - \frac{1}{2}\left\langle \int \lambda \cdot dM \right\rangle_t\right) \\ &= \exp\left(-\int_0^t \lambda_s \cdot dM_s - \frac{1}{2}K_t\right). \end{aligned}$$

[9] $X = X^c + X^d$, where X^c is the continuous part and X^d is the discontinuous part.

From Theorem II.2 in [Lepingle and Mèmin, 1978] it follows that Z is square integrable, since in this case we have that $\int \lambda \cdot dM$ is a square integrable local martingale null at zero and $\left\langle \int \lambda \cdot dM \right\rangle$ is bounded. Further, the **Novikov Condition**

$$\mathbb{E}\left(\exp\left(\frac{1}{2}K_t\right)\right) < \infty \quad \forall\, t \geq 0,$$

is fulfilled, which yields that Z is even a square integrable martingale with respect to $\mathbb{P}$. Hence, the density

$$\frac{d\mathbb{Q}_T}{d\mathbb{P}} := Z_T = \mathcal{E}\left(-\int \lambda \cdot dM\right)_T \tag{2.2.25}$$

defines a probability measure $\mathbb{Q}_T$ equivalent to $\mathbb{P}$. Note, that we choose a càdlàg version of the martingale $Z_t = \mathbb{E}(Z_T \,|\, \mathcal{F}_t)$. The following definition and theorem are based on [Föllmer and Schweizer, 1990].

Definition 2.12 (Minimal martingale measure). *We call an equivalent martingale measure $\mathbb{Q}$ with square integrable density* ***minimal****, if $\mathbb{Q} = \mathbb{P}$ on $\mathcal{F}_0$ and if it holds for any $L \in \mathcal{M}^2(\mathbb{P})$ with $\langle L, M^i \rangle = 0$ for each i, that L is a martingale under $\mathbb{Q}$.*

Under $\mathbb{P}$, $\langle L, M^i \rangle = 0$ is equivalent to L beeing strongly orthogonal to M^i, since we know that $(L_t M_t^i - \langle L, M^i \rangle_t)_{0 \leq t \leq T}$ is a local $\mathbb{P}$-martingale.

Remark. Looking at the Föllmer Schweizer decomposition, the probability measure $\mathbb{Q}_T$ is minimal in the sense that we maintain the orthogonality property of L to the stochastic integral as the next theorem shows. Thus, we disturb the structure of X as little as possible. We additionally need, that the density Z is $\mathbb{P}$-square-integrable to ensure that the claim H is $\mathbb{Q}_T$-integrable.

Theorem 2.18. *Suppose X is continuous.*

(i) The minimal martingale measure $\mathbb{Q}_T$ defined by (2.2.25) *is unique.*

(ii) The minimal martingale measure $\mathbb{Q}_T$ exists, if and only if

$$Z_t = \exp\left(-\int_0^t \lambda_s \cdot dM_s - \frac{1}{2}K_t\right) \quad (0 \leq t \leq T)$$

is in $\mathcal{M}^2(\mathbb{P})$. In this case $\mathbb{Q}_T$ is defined by (2.2.25).

(iii) If $L \in \mathcal{M}^2(\mathbb{P})$ with $\langle L, M^i \rangle = 0$ under $\mathbb{P}$ for each i, then it holds $\langle L, X^i \rangle = 0$ under $\mathbb{Q}_T$ for each i.

Proof. Ad (i): Let $\mathbb{Q}_T^* \sim \mathbb{P}$ be another minimal martingale measure defined by a process $Z^* = (Z_t^*)_{0 \leq t \leq T} \in \mathcal{M}^2(\mathbb{P})$. Then Z_t^* has a Galtchouk-Kunita-Watanabe decomposition with respect to M under $\mathbb{P}$

$$Z_t^* = \mathbb{E}(Z_t^*) + \int_0^t \beta_s \cdot dM_s + L_t \quad \mathbb{P}\text{-a.s.},$$

where $\beta \in L^2(M)$[10] and $L \in \mathcal{M}^2(\mathbb{P})$, with $\mathbb{E}(L_0) = 0$, is strongly orthogonal to M^i for each i. Applying the product rule to Z^* and $X^i = X_0^i + M^i + A^i$ yields

$$d(Z^* X^i) = \left(X_-^i \, dZ^* + Z_-^* \, dM^i + d\langle Z^*, A^i\rangle\right) + Z_-^* \, dA^i + d\langle Z^*, M\rangle.$$

The product $Z^* X^i$ is a $\mathbb{P}$-martingale. From [Protter, 2005] we know that $\langle Z^*, A^i\rangle$ is a $\mathbb{P}$-martingale, since Z^* is a $\mathbb{P}$-martingale and A is a previsible process of bounded variation. Thus, the bracket term on the right hand side is the differential of a $\mathbb{P}$-martingale and we obtain for A^i the representation

$$A_t^i = \int_0^t -\frac{1}{Z_{s-}^*} \, d\langle Z^*, M^i\rangle_s = \left(\int_0^t -\frac{1}{Z_{s-}^*} \beta_s^{tr} \, d\langle M\rangle_s\right)^i.$$

Since X is continuous and Z^* satisfies the appropriate conditions, we get from Theorem 2.17 that the structure condition is fulfilled and from the calculation after the definition of the structure condition we conclude that λ is given by

$$\lambda = -\frac{\beta}{Z_-^*}.$$

$\mathbb{Q}_T^* \sim \mathbb{P}$ is equivalent to $Z^* > 0$ $\mathbb{P}$-a.s.. Thus, we get from $\beta \in L^2(M)$ that $\lambda \in L^2(M)$. Now we use that $\mathbb{Q}_T^*$ is minimal. From $\mathbb{Q}_T^* = \mathbb{P}$ on $\mathcal{F}_0$ we get $Z_0^* = 1$ and hence also $\mathbb{E}(Z_t^*) = \mathbb{E}(Z_0^*) = 1$. Further, since $L \in \mathcal{M}^2(\mathbb{P})$ is strongly orthogonal to each M^i we get that L is a martingale under $\mathbb{Q}_T^*$. Hence we have

$$\langle L\rangle^{\mathbb{Q}_T^*} = \langle L, Z^*\rangle^{\mathbb{P}} = 0,$$

which implies $L \equiv 0$. Hence, Z^* has the simplified decomposition

$$Z_t^* = 1 - \int_0^t Z_{s-}^* \lambda_s \cdot dM_s \quad \mathbb{P}\text{-a.s.}$$

and since M is continuous we get $Z^* = Z$.
Ad (ii): With Theorem 2.17 we have that the process Z is is well defined. But, as we have seen, it is in general only a locally square integrable local $\mathbb{P}$-martingale. If $\mathbb{Q}_T$ exists and is defined by (2.2.25) then Z is by definition a square integrable $\mathbb{P}$-martingale. Conversely, suppose $Z \in \mathcal{M}^2(\mathbb{P})$. We will show that the corresponding martingale measure $\mathbb{Q}_T$ is minimal. Let $L \in \mathcal{M}^2(\mathbb{P})$ with $\langle L, M^i\rangle = 0$ under $\mathbb{P}$ for each i. Since Z solves (2.2.24) it holds $\langle L, Z\rangle = 0$. Thus, ZL is a local $\mathbb{P}$-martingale and hence L is a local $\mathbb{Q}_T$-martingale. Since $L \in \mathcal{M}^2(\mathbb{P})$ we have

$$\sup_{0 \le t \le T} |L_t| \in L^2(\mathcal{F}_T, \mathbb{P})$$

[10] See Lemma 2.2.

and with $Z_T \in L^2(\mathcal{F}_T, \mathbb{P})$ we obtain with Cauchy-Schwarz' Inequality

$$\sup_{0 \le t \le T} |L_t| \in L^1(\mathcal{F}_T, \mathbb{Q}_T).$$

Thus, L is even a $\mathbb{Q}_T$-martingale.
Ad (iii): Again, let $L \in \mathcal{M}^2(\mathbb{P})$ with $\langle L, M^i \rangle = 0$ under $\mathbb{P}$ for each i. We need to show $\langle L, X^i \rangle = 0$ under $\mathbb{Q}_T$ for each i. Since X^i is continuous and A^i is a continuous finite variation process we obtain with (2.2.23)

$$\begin{aligned} \langle L, X^i \rangle &= \langle L^c, X^i \rangle \\ &= [L, X^i] \\ &= [L, M^i] + [L, A^i] \\ &= [L, M^i] \end{aligned}$$

under $\mathbb{Q}_T$. Under $\mathbb{P}$ we further have, since M^i is continuous,

$$[L, M^i] = \langle L^c, M^i \rangle = \langle L, M^i \rangle = 0.$$

But since $\mathbb{Q}_T \ll \mathbb{P}$ we even have $[L, M^i] = 0$ under $\mathbb{Q}_T$, which completes the proof. □

Lemma 2.19. *If ϕ^H is a pseudo-optimal L^2-strategy for H and Z is the solution of* (2.2.24)*, then $ZV(\phi^H)$ is a local martingale with respect to $\mathbb{P}$.*

Proof. Look at the decomposition (2.2.14)

$$V_t(\phi^H) = \mathbb{E}(H) + \int_0^t \xi_s^H \cdot dX_s + L_t^H \quad \mathbb{P}\text{-a.s.},$$

where $\xi^H \in \Xi$ and $L^H \in \mathcal{M}^2(\mathbb{P})$, with $\mathbb{E}(L_0^H) = 0$, is strongly orthogonal to $\mathcal{I}^2(M)$ with respect to $\mathbb{P}$. We know that the product ZX^i is a local $\mathbb{P}$-martingale for $i = 1, \dots, d$. Since X is continuous, $Z \int \xi \, dX$ is a local $\mathbb{P}$-martingale for $\xi \in \Xi$ as well. Further, we have that ZL^H is a local $\mathbb{P}$-martingale. In total we get

$$Z_t V_t(\phi^H) = Z_t \mathbb{E}(H) + Z_t \int_0^t \xi_s^H \cdot dX_s + Z_t L_t^H \quad (0 \le t \le T),$$

which is a local $\mathbb{P}$-martingale as composition of local $\mathbb{P}$-martingales. □

Consider the situation of Lemma 2.19 where Z is a square-integrable $\mathbb{P}$-martingale. We know that $\sup_{0 \le t \le T} |V_t(\phi^H)|$ is square-integrable with respect to $\mathbb{P}$. Using Cauchy-Schwarz' inequality we obtain $ZV(\phi^H) \in L^1(\mathbb{P})$. Thus, $ZV(\phi^H)$ is even a true $\mathbb{P}$-martingale, which is equivalent to $V(\phi^H)$ beeing a $\mathbb{Q}_T$-martingale. Therefore, we define

$$V_t^{H,\mathbb{Q}_T} := \mathbb{E}_{\mathbb{Q}_T}(H \mid \mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}_T}(V_T(\phi^H) \mid \mathcal{F}_t) = V_t(\phi^H) \quad (0 \le t \le T).$$

Note, that $V_t^{H,\mathbb{Q}_T}$ is well defined since

$$\mathbb{E}_{\mathbb{Q}_T}(H \mid \mathcal{F}_t) = \frac{1}{Z_t}\mathbb{E}(HZ_T \mid \mathcal{F}_t)$$

and H and Z_T are both square-integrable with respect to $\mathbb{P}$. Hence, by Cauchy-Schwarz we have $H \in L^1(\mathbb{Q}_T)$.
Further, X is a local $\mathbb{Q}_T$-martingale. Thus, $V_t^{H,\mathbb{Q}_T}$ has a Galtchouk-Kunita-Watanabe decomposition with respect to X under $\mathbb{Q}_T$ and we get for $0 \le t \le T$

$$\begin{aligned} V_t^{H,\mathbb{Q}_T} &= \mathbb{E}_{\mathbb{Q}_T}\left(V_t^{H,\mathbb{Q}_T}\right) + \int_0^t \xi_s^{H,\mathbb{Q}_T} \cdot dX_s + L_t^{H,\mathbb{Q}_T} \\ &= \mathbb{E}_{\mathbb{Q}_T}(H) + \int_0^t \xi_s^{H,\mathbb{Q}_T} \cdot dX_s + L_t^{H,\mathbb{Q}_T} \quad \mathbb{Q}_T\text{-a.s.}, \end{aligned}$$

where $\xi^{H,\mathbb{Q}_T} \in L^2(X)$[11] and $L^{H,\mathbb{Q}_T} \in \mathcal{M}^2(\mathbb{Q}_T)$ with $\mathbb{E}_{\mathbb{Q}_T}(L_0^{H,\mathbb{Q}_T}) = 0$ is strongly $\mathbb{Q}_T$-orthogonal to X^i for each i. By switching to the $\mathbb{P}$-measure we obtain

$$V_t^{H,\mathbb{Q}_T} = \mathbb{E}(Z_T H) + \int_0^t \xi_s^{H,\mathbb{Q}_T} \cdot dX_s + L_t^{H,\mathbb{Q}_T} \quad \mathbb{P}\text{-a.s.},$$

where $\xi^{H,\mathbb{Q}_T} \in \Xi$ and $L^{H,\mathbb{Q}_T} \in \mathcal{M}^2(\mathbb{P})$, with $\mathbb{E}(Z_T L_0^{H,\mathbb{Q}_T}) = 0$, is by Theorem 2.18 strongly $\mathbb{P}$-orthogonal to M^i for each i.

In summary we get the following theorem.

Theorem 2.20. *Assume X is a continuous semimartingale. Define $\mathbb{Q}_T$, $V^{H,\mathbb{Q}_T}$ as above and suppose $Z_T = \mathcal{E}\left(-\int \lambda \cdot dM\right) \in \mathcal{M}^2(\mathbb{P})$. If either H admits a Föllmer-Schweizer decomposition or*

$$V_t^{H,\mathbb{Q}_T} = \mathbb{E}(Z_T H) + \int_0^t \xi_s^{H,\mathbb{Q}_T} \cdot dX_s + L_t^{H,\mathbb{Q}_T} \quad \mathbb{P}\text{-a.s.}, \ (0 \le t \le T), \tag{2.2.26}$$

where $\xi^{H,\mathbb{Q}_T} \in \Xi$ and $L^{H,\mathbb{Q}_T} \in \mathcal{M}^2(\mathbb{P})$, with $\mathbb{E}(Z_T L_0^{H,\mathbb{Q}_T}) = 0$, is strongly orthogonal to $\mathcal{I}^2(M)$, then $V_T^{H,\mathbb{Q}_T}$ gives a Föllmer-Schweizer-decomposition of H and $\xi^{H,\mathbb{Q}_T}$ is a pseudo-optimal L^2-strategy for H. K beeing uniformly bounded is a sufficient condition for the square integrability of Z_T and the uniqueness of the Föllmer-Schweizer decomposition.

Proof. Since $\mathbb{Q}_T = \mathbb{P}$ on $\mathcal{F}_0$ we conclude that the starting values of the decompositions (2.2.13) and (2.2.26) of H and $V_T^{H,\mathbb{Q}_T}$, respectively, must coincide. If H admits a Föllmer-Schweizer decomposition, then Theorem 2.18 implies that L^H is a local $\mathbb{Q}_T$-martingale, which is strongly $\mathbb{Q}_T$-orthogonal to $\mathcal{I}^2(X)$. Since the Galtchouk-Kunita-Watanabe decomposition is unique, we conclude that (2.2.13) and (2.2.26) for $t = T$

[11] Cp. Lemma 2.2 under the measure $\mathbb{Q}_T$.

must be equal.
If we have (2.2.26), then the last argument before this theorem shows that for $t = T$ we obtain a Föllmer-Schweizer decomposition of H, which by uniqueness must again be the same as (2.2.13).
With Theorem 2.12 we conclude that $\xi^{H,\mathbb{Q}_T}$ gives a pseudo-optimal L^2-strategy for H and that uniform boundedness of K is sufficient follows from Theorem 2.16 and from Theorem II.2 [Lepingle and Mèmin, 1978]. □

For continuous X, a locally risk-minimizing strategy can be obtained by finding the Galtchouk-Kunita-Watanabe decomposition of H under the minimal martingale measure $\mathbb{Q}_T$. This is useful, since the density process Z_T of $\mathbb{Q}_T$ is explicitly given and we can study the behaviour of X under $\mathbb{Q}_T$.

Remark. As a byproduct we obtain from (2.2.26) that $V_t(\phi^H)$ can be used for pricing H at time t, where ϕ^H is a H-pseudo-optimal L^2-strategy. In this sense

$$V_t^{H,\mathbb{Q}_T} = \mathbb{E}_{\mathbb{Q}_T}(H \mid \mathcal{F}_t)$$

can be interpreted as *intrinsic valuation process* for H and $\mathbb{Q}_T$ is the corresponding valuational operator. But we have to keep in mind that $V_t^{H,\mathbb{Q}_T}$ is only a valuation with respect to our definition (2.2.3) of local risk-minimization.

With the help of the minimal martingale measure we can now prove a generalization of the Föllmer-Schweizer decomposition based on [Schweizer, 1994]. This theorem will specifically come in handy in the next section. For the proof recall the following two inequalities:

(i) **Burkholder-Davis-Gundy's Inequality:** Suppose M is a local $\mathbb{P}$-martingale with $M_0 = 0$. Then for any $1 \leq r < \infty$ there $\exists$ positive constants c_r, C_r, such that $\forall\, t > 0$ we have

$$c_r\, \mathbb{E}\left([M]_t^{r/2}\right) \leq \mathbb{E}\left(\left(\sup_{s\leq t} |M_s|\right)^r\right) \leq C_r\, \mathbb{E}\left([M]_t^{r/2}\right).$$

(ii) **Doob's Maximal Inequality:** Suppose V is a càdlàg $\mathbb{Q}$-martingale and $t > 0$. Then for every $q > 1$ we have

$$\mathbb{E}_{\mathbb{Q}}\left(\left(\sup_{s\leq t} |V_s|\right)^q\right) \leq \left(\frac{q}{q-1}\right)^q \mathbb{E}_{\mathbb{Q}}\left(V_t^q\right).$$

Theorem 2.21. *Suppose X is continuous and K is uniformly bounded. Then any $H \in L^p(\mathcal{F}_T, \mathbb{P})$ with $p \geq 2$ admits a Föllmer-Schweizer decomposition with $\xi^H \in L^r(M)$ and $L^H \in \mathcal{M}^r(\mathbb{P})$ for every $r < p$.*

Proof. With Theorem 2.17 we know that X satisfies the structure condition. Hence, by Theorem 2.16 we know that $H \in L^p(\mathcal{F}_T, \mathbb{P})$, for some $p \geq 2$, admits a Föllmer-Schweizer decomposition. With $\mathbb{E}(H \mid \mathcal{F}_0) = \mathbb{E}(H) + L_0^H$ we automatically get $L_0^H \in$

$L^p(\mathbb{P})$. It remains to prove $\xi^H \in L^r(M)$ and $L^H - L^H_0 = \bar{L}^H \in \mathcal{M}^r_0(\mathbb{P})$ for every $r < p$. Since X is continuous and K is uniformly bounded, we know from Theorem II.2 [Lepingle and Mèmin, 1978] and Theorem 2.18 that

$$Z = \mathcal{E}\left(-\int \lambda \cdot dM\right)$$

is in $\mathcal{M}^2(\mathbb{P})$ and defines the minimal martingale measure. Since K is uniformly bounded, we have for every $1 \le r < \infty$

$$Z_T \in \mathcal{M}^r(\mathbb{P}), \tag{2.2.27}$$

$$1/Z_T \in \mathcal{M}^r(\mathbb{Q}_T). \tag{2.2.28}$$

Continuity of X and Theorem 2.18 further imply with $\bar{V}^{H,\mathbb{Q}_T} = V^{H,\mathbb{Q}_T} - V_0^{H,\mathbb{Q}_T}$

$$\left\langle \int \xi^H \cdot dM \right\rangle_T = \left[\int \xi^H \cdot dM\right]_T = \left[\int \xi^H \cdot dX\right]_T \le \left[V^{H,\mathbb{Q}_T}\right]_T = \left[\bar{V}^{H,\mathbb{Q}_T}\right]_T$$

and

$$\left[\bar{L}^H\right]_T \le \left[V^{H,\mathbb{Q}_T}\right]_T = 0 \left[\bar{V}^{H,\mathbb{Q}_T}\right]_T .$$

But since $\bar{L}^H$ is a local $\mathbb{P}$-martingale starting in zero, we obtain by Burkholder-Davis-Gundy's Inequality for every $r \ge 1$

$$\mathbb{E}\left(\left(\sup_{0\le t\le T} \left|\bar{L}^H_t\right|\right)^r\right) \le C_r\, \mathbb{E}\left(\left[\bar{L}^H\right]_T^{r/2}\right),$$

where C_r is a constant depending on r. Let us now prove

$$\left[\bar{V}^{H,\mathbb{Q}_T}\right]_T \in L^{r/2}(\mathbb{P}) \quad \text{for every } r < p.$$

Then the integrability conditions on ξ^H and $\bar{L}^H$ are an immediate consequence. Since $Z\bar{V}^{H,\mathbb{Q}_T}$ is a $\mathbb{P}$-martingale, we know $\bar{V}^{H,\mathbb{Q}_T}$ is a $\mathbb{Q}_T$-martingale. Hence, we obtain by Burkholder-Davis-Gundy's Inequality and Doob's Maximal Inequality for $2s > 1$

$$\mathbb{E}_{\mathbb{Q}_T}\left(\left[\bar{V}^{H,\mathbb{Q}_T}\right]_T^s\right) \le \frac{1}{c_s}\, \mathbb{E}_{\mathbb{Q}_T}\left(\left(\sup_{0\le t\le T}\left|\bar{V}_t^{H,\mathbb{Q}_T}\right|\right)^{2s}\right) \le C_s\, \mathbb{E}_{\mathbb{Q}_T}\left(\left|\bar{V}_T^{H,\mathbb{Q}_T}\right|^{2s}\right),$$

where c_s, C_s are constants depending on s. By assumption we have $V_T^{H,\mathbb{Q}_T} = H \in L^p(\mathcal{F}_T, \mathbb{P})$ for some $p \ge 2$. Consequently $\bar{V}_T^{H,\mathbb{Q}_T} \in L^p(\mathbb{P})$ and with (2.2.27) and Hölder's Inequality we have $\bar{V}_T^{H,\mathbb{Q}_T} \in L^{2s}(\mathbb{Q}_T)$ for $2s < p$ and using the above inequality implies that $\left[\bar{V}^{H,\mathbb{Q}_T}\right]_T \in L^s(\mathbb{Q}_T)$ for every $s < p/2$. Finally, with (2.2.28) and Hölder's Inequality we conclude $\left[\bar{V}^{H,\mathbb{Q}_T}\right]_T \in L^{r/2}(\mathbb{P})$ for every $r < p$. □

Remark. In Lemma 6 of [Pham et al., 1996] it is even proven $\xi^H \in L^p(M)$ and $L^H \in \mathcal{M}^p(\mathbb{P})$. Instead of switching to the $\mathbb{Q}_T$-measure they work directly under $\mathbb{P}$ and thus, do not lose the case $r = p$.

2.3 Mean-variance hedging

The following section is based on [Monat and Stricker, 1995]. From now on, if not mentioned otherwise, we do not assume continuity of X.
In comparison to local risk-minimization we do not rely on the terminal constraint $V_T = H$ $\mathbb{P}$-a.s.. Instead we focus on self-financing trading strategies and minimize the hedging error at maturity.

Definition 2.13 (mean-variance optimal). *Let $H \in L^2(\mathcal{F}_T, \mathbb{P})$, then the unique solution $\left(V_0^H, \xi^H\right) \in \mathbb{R} \times \Xi$ of*

$$\min_{(V_0,\xi)\in\mathbb{R}\times\Xi} \mathbb{E}\left(\left(H - V_0 - \int_0^T \xi_s \cdot dX_s\right)^2\right)$$

is called ***mean-variance optimal****. Similarly, for any given initial capital $V_0 \in \mathbb{R}$ the unique solution $\xi^{V_0} \in \Xi$ of*

$$\min_{\xi\in\Xi} \mathbb{E}\left(\left(H - V_0 - \int_0^T \xi_s \cdot dX_s\right)^2\right).$$

is called ***mean-variance optimal****.*

To find its unique solution we want to use Hilbert's Projection Theorem. Thus, we project H onto the linear space spanned by $\mathbb{R}$ and

$$G_T(\Xi) := \left\{ \int_0^T \xi_s \cdot dX_s \,\middle|\, \xi \in \Xi \right\}.$$

This projection idea only works if $\{\mathbb{R} + G_T(\Xi)\}$ is a closed subspace of $L^2(\mathbb{P})$. The next subsection proves this condition under the assumption that X satisfies the structure condition and that the mean-variance tradeoff process K is uniformly bounded.

2.3.1 Closedness of $G_T(\Xi)$

Before we can prove the closedness of $G_T(\Xi)$ we have to prove the continuity of the Föllmer-Schweizer decomposition. As for the existence and uniqueness Theorem 2.16 of the Föllmer-Schweizer decomposition, we need two auxiliary results. Note again that we switch to the different formulation of the Föllmer-Schweizer decomposition (Compare Equation 2.1.4).

Lemma 2.22. *Suppose X satisfies the structure condition. Let $0 \le T_1 \le T_2 \le T$ be previsible stopping times and assume there $\exists\, b \in (0,1)$, such that $K_{T_2-} - K_{T_1} \le b$*

$\mathbb{P}$*-a.s.. Further, suppose* $H, H^p \in L^2(\Omega, \mathcal{F}_{T_2-}, \mathbb{P})$ *with decompositions*

$$H = \hat{H} + \int_0^T \mathbb{I}_{(T_1,T_2)}(s)\xi_s \cdot dX_s + \int_0^T \mathbb{I}_{(T_1,T_2)}(s)\, d\hat{L}_s \quad \mathbb{P}\text{-a.s.},$$
$$H^p = \hat{H}^p + \int_0^T \mathbb{I}_{(T_1,T_2)}(s)\xi_s^p \cdot dX_s + \int_0^T \mathbb{I}_{(T_1,T_2)}(s)\, d\hat{L}_s^p \quad \mathbb{P}\text{-a.s.},$$

as defined in Equation (2.2.15)*. If we have*

$$H^p \xrightarrow[p\to\infty]{L^2} H,$$

then it holds

$$\hat{H}^p \xrightarrow[p\to\infty]{L^2} \hat{H},$$
$$\mathbb{I}_{(T_1,T_2)}\hat{\xi}^p \xrightarrow[p\to\infty]{L^2(M)} \mathbb{I}_{(T_1,T_2)}\hat{\xi},$$
$$\int_0^T \mathbb{I}_{(T_1,T_2)}(s)\, d\hat{L}_s^p \xrightarrow[p\to\infty]{L^2} \int_0^T \mathbb{I}_{(T_1,T_2)}(s)\, d\hat{L}_s.$$

Proof. Let $\hat{\xi}$ and $\hat{\xi}^p$ be the unique fixed points of the mappings Ψ_H and Ψ_{H^p}, respectively, which are defined in Definition 2.11. Since Ψ_{H^p} is a contraction[12] with parameter $\sqrt{b}$ independent of p, we have

$$\begin{aligned}
\|\hat{\xi}^p - \hat{\xi}\|_{L^2(M)} &= \|\Psi_{H^p}(\hat{\xi}^p) - \Psi_H(\hat{\xi})\|_{L^2(M)} \\
&\le \|\Psi_{H^p}(\hat{\xi}^p) - \Psi_{H^p}(\hat{\xi})\|_{L^2(M)} + \|\Psi_{H^p}(\hat{\xi}) - \Psi_H(\hat{\xi})\|_{L^2(M)} \\
&\le \sqrt{b}\|\hat{\xi}^p - \hat{\xi}\|_{L^2(M)} + \|\Psi_{H^p}(\hat{\xi}) - \Psi_H(\hat{\xi})\|_{L^2(M)},
\end{aligned}$$

which is equivalent to

$$\|\hat{\xi}^p - \hat{\xi}\|_{L^2(M)} \le \frac{1}{1-\sqrt{b}}\|\Psi_{H^p}(\hat{\xi}) - \Psi_H(\hat{\xi})\|_{L^2(M)}. \tag{2.3.1}$$

We will now show continuity in H of Ψ_H with respect to $\|\cdot\|_{L^2(M)}$ for a general but fixed $\theta \in L^2(M)$.

Denote $\hat{\theta}^p = \Psi_{H^p}(\theta)$ and $\hat{\theta} = \Psi_H(\theta)$, then we obtain with the equation before Equation (2.2.15)

$$H - \int_0^T \theta_s^{tr}\, dA_s = \hat{H} + \int_0^T \mathbb{I}_{(T_1,T_2)}(s)\hat{\theta}_s \cdot dM_s + \int_0^T \mathbb{I}_{(T_1,T_2)}(s)\, d\hat{L}_s \quad \mathbb{P}\text{-a.s.}$$

[12] Compare Lemma 2.14.

and

$$H^p - \int_0^T \theta_s^{tr}\, dA_s = \hat{H}^p + \int_0^T \mathbb{I}_{(T_1,T_2)}(s)\hat{\theta}_s^p \cdot dM_s + \int_0^T \mathbb{I}_{(T_1,T_2)}(s)\, d\hat{L}_s^p \quad \mathbb{P}\text{-a.s.}.$$

Thus, it holds

$$\|H^p - H\|_2 \geq \left\| \int_0^T \mathbb{I}_{(T_1,T_2)}(s) \left(\hat{\theta}_s^p - \hat{\theta}_s\right) \cdot dM_s \right\|_2,$$

where we used the orthogonality of the three terms. Under the assumption

$$H^p \xrightarrow[p\to\infty]{L^2} H$$

we obtain

$$\int_0^T \mathbb{I}_{(T_1,T_2)}(s)\hat{\theta}_s^p \cdot dM_s \xrightarrow[p\to\infty]{L^2} \int_0^T \mathbb{I}_{(T_1,T_2)}(s)\hat{\theta}_s \cdot dM_s$$

and finally, with Equation (2.3.1) also

$$\int_0^T \mathbb{I}_{(T_1,T_2)}(s)\hat{\xi}_s^p \cdot dM_s \xrightarrow[p\to\infty]{L^2} \int_0^T \mathbb{I}_{(T_1,T_2)}(s)\hat{\xi}_s \cdot dM_s.$$

Since X satisfies the structure condition and $K_{T_2-} - K_{T_1} \leq b$, we further have

$$\begin{aligned}
\left\| \int_0^T \left(\hat{\xi}_s^p - \hat{\xi}_s\right)^{tr} dA_s \right\|_2^2 &= \left\| \int_0^T \left(\hat{\xi}_s^p - \hat{\xi}_s\right)^{tr} \gamma_s\, dW_s \right\|_2^2 \\
&= \left\| \int_0^T \left(\hat{\xi}_s^p - \hat{\xi}_s\right)^{tr} \sigma_s \lambda_s\, dW_s \right\|_2^2 \\
&\leq \|K_{T_2-} - K_{T_1}\|_\infty \mathbb{E}\left(\int_0^T \left(\hat{\xi}_s^p - \hat{\xi}_s\right)^{tr} \sigma_s \left(\hat{\xi}_s^p - \hat{\xi}_s\right)\, dW_s \right) \\
&\leq b \left\| \int_0^T \left(\hat{\xi}_s^p - \hat{\xi}_s\right) \cdot dM_s \right\|_2^2.
\end{aligned}$$

Thus, we also have

$$\int_0^T \hat{\xi}_s^{p\,tr} dA_s \xrightarrow[p\to\infty]{L^2} \int_0^T \hat{\xi}_s^{tr} dA_s,$$

which implies

$$\left\| \left(H^p - \int_0^T \hat{\xi}_s^{p\,tr} dA_s\right) - \left(H - \int_0^T \hat{\xi}_s^{tr} dA_s\right) \right\|_2^2 \xrightarrow[p\to\infty]{L^2} 0.$$

By assumption we have from Equation (2.2.15)

$$H - \int_0^T \hat{\xi}_s^{tr} dA_s = \hat{H} + \int_0^T \mathbb{I}_{(T_1,T_2)}(s)\hat{\xi}_s \cdot dM_s + \int_0^T \mathbb{I}_{(T_1,T_2)}(s)\, d\hat{L}_s \quad \mathbb{P}\text{-a.s.},$$
$$H^p - \int_0^T \hat{\xi}_s^{p\,tr} dA_s = \hat{H}^p + \int_0^T \mathbb{I}_{(T_1,T_2)}(s)\hat{\xi}_s^p \cdot dM_s + \int_0^T \mathbb{I}_{(T_1,T_2)}(s)\, d\hat{L}_s^p \quad \mathbb{P}\text{-a.s.}.$$

Since the three terms

$$\hat{H}^p - \hat{H}, \quad \int_0^T \mathbb{I}_{(T_1,T_2)}(s)\left(\hat{\xi}_s^p - \hat{\xi}_s\right) \cdot dM_s, \quad \int_0^T \mathbb{I}_{(T_1,T_2)}(s)\, d\hat{L}_s^p - \int_0^T \mathbb{I}_{(T_1,T_2)}(s)\, d\hat{L}_s$$

are orthogonal, the last L^2-convergence implies

$$\hat{H}^p \xrightarrow[p\to\infty]{L^2} \hat{H},$$
$$\int_0^T \mathbb{I}_{(T_1,T_2)}(s)\, d\hat{L}_s^p \xrightarrow[p\to\infty]{L^2} \int_0^T \mathbb{I}_{(T_1,T_2)}(s)\, d\hat{L}_s.$$

□

Lemma 2.23. *Suppose X satisfies the structure condition and K is uniformly bounded. Let $0 \le T_0 \le T$ be a previsible stopping time and assume we have $H, H^p \in L^2(\mathcal{F}_{T_0}, \mathbb{P})$ with*

$$H = \tilde{H} + \int_0^T \mathbb{I}_{T_0}(s)\tilde{\xi}_s \cdot dX_s + \tilde{L}_{T_0} \quad \mathbb{P}\text{-}a.s.,$$
$$H^p = \tilde{H}^p + \int_0^T \mathbb{I}_{T_0}(s)\tilde{\xi}_s^p \cdot dX_s + \tilde{L}_{T_0}^p \quad \mathbb{P}\text{-}a.s.,$$

as defined in Equation (2.2.16) *of Lemma 2.15. If we have*

$$H^p \xrightarrow[p\to\infty]{L^2} H,$$

then it holds

$$\tilde{H}^p \xrightarrow[p\to\infty]{L^2} \tilde{H},$$
$$\mathbb{I}_{T_0}\tilde{\xi}^p \xrightarrow[p\to\infty]{L^2(M)} \mathbb{I}_{T_0}\tilde{\xi},$$
$$\tilde{L}_{T_0}^p \xrightarrow[p\to\infty]{L^2} \tilde{L}_{T_0}.$$

Proof. Under the assumption $H^p \xrightarrow[p\to\infty]{L^2} H$ we have with the use of the tower property and Jensen's Inequality

$$\mathbb{E}(H^p \,|\, \mathcal{F}_{T_0-}) \xrightarrow[p\to\infty]{L^2} \mathbb{E}(H \,|\, \mathcal{F}_{T_0-}).$$

Substraction yields

$$H^p - \mathbb{E}(H^p \,|\, \mathcal{F}_{T_0-}) \xrightarrow[p\to\infty]{L^2} H - \mathbb{E}(H \,|\, \mathcal{F}_{T_0-})$$

and inserting the given decompositions we obtain

$$\int_0^T \mathbb{I}_{T_0}(s)\tilde{\xi}_s^p \cdot dM_s + \tilde{L}_{T_0}^p \xrightarrow[p\to\infty]{L^2} \int_0^T \mathbb{I}_{T_0}(s)\tilde{\xi}_s \cdot dM_s + \tilde{L}_{T_0}.$$

However, the unique Galtchouk-Kunita-Watanabe decompositions of $H^p - \mathbb{E}(H^p \,|\, \mathcal{F}_{T_0-})$ and $H - \mathbb{E}(H \,|\, \mathcal{F}_{T_0-})$ are given by

$$\begin{aligned} H^p - \mathbb{E}(H^p \,|\, \mathcal{F}_{T_0-}) &= \int_0^T \mathbb{I}_{T_0}(s)\tilde{\xi}_s^p \cdot dM_s + \tilde{L}_{T_0}^p \quad \mathbb{P}\text{-a.s.}, \\ H - \mathbb{E}(H \,|\, \mathcal{F}_{T_0-}) &= \int_0^T \mathbb{I}_{T_0}(s)\tilde{\xi}_s \cdot dM_s + \tilde{L}_{T_0} \quad \mathbb{P}\text{-a.s..} \end{aligned}$$

Since the Galtchouk-Kunita-Watanabe decomposition is a projection on a closed subspace of L^2, it is continuous in the L^2 sense. Thus, we obtain

$$\begin{aligned} \int_0^T \mathbb{I}_{T_0}(s)\tilde{\xi}_s^p \cdot dM_s &\xrightarrow[p\to\infty]{L^2} \int_0^T \mathbb{I}_{T_0}(s)\tilde{\xi}_s \cdot dM_s, \\ \tilde{L}_{T_0}^p &\xrightarrow[p\to\infty]{L^2} \tilde{L}_{T_0}. \end{aligned}$$

In general, if $\xi^p \xrightarrow[p\to\infty]{L^2(M)} \xi$ for some $\xi^p, \xi \in \Xi$, then it also holds $\int_0^T \xi_s^p \cdot dX_s \xrightarrow[p\to\infty]{L^2} \int_0^T \xi_s \cdot dX_s$. This is true since

$$\begin{aligned} \left\| \int_0^T (\xi_s^p - \xi_s) \cdot dX_s \right\|_2 &\le \|\xi^p - \xi\|_{L^2(M)} + \left\| \int_0^T (\xi_s^p - \xi_s)^{tr} \, dA_s \right\|_2 \\ &\le \|\xi^p - \xi\|_{L^2(M)} + \left\| \int_0^T (\xi_s^p - \xi_s)^{tr} \, \gamma_s \, dW_s \right\|_2 \\ &\le \|\xi^p - \xi\|_{L^2(M)} + \left\| \int_0^T (\xi_s^p - \xi_s)^{tr} \, \sigma_s \lambda_s \, dW_s \right\|_2 \\ &\le \left(1 + \|K\|_\infty^{1/2}\right) \|\xi^p - \xi\|_{L^2(M)}. \end{aligned}$$

Therefore, in our setting we conclude $\int_0^T \mathbb{I}_{T_0}(s)\tilde{\xi}_s^p \cdot dX_s \xrightarrow[p\to\infty]{L^2} \int_0^T \mathbb{I}_{T_0}(s)\tilde{\xi}_s \cdot dX_s$ and consequently, $\tilde{H}^p \xrightarrow[p\to\infty]{L^2} \tilde{H}$. $\square$

Now use the same notations as in the proof of Theorem 2.16 and set

$$\begin{aligned}
\xi &= \sum_{n=1}^{N} \left(\mathbb{I}_{(T_{n-1},T_n)} \hat{\xi}^{n-1} + \mathbb{I}_{T_n} \tilde{\xi}^{n} \right), \\
\xi^p &= \sum_{n=1}^{N} \left(\mathbb{I}_{(T_{n-1},T_n)} \hat{\xi}^{p,n-1} + \mathbb{I}_{T_n} \tilde{\xi}^{p,n} \right), \\
L_T &= \hat{L}_0 + \sum_{n=1}^{N} \left(\tilde{L}^n_{T_n} + \int_0^T \mathbb{I}_{(T_{n-1},T_n)}(s)\, d\hat{L}^{n-1}_s \right), \\
L^p_T &= \hat{L}^p_0 + \sum_{n=1}^{N} \left(\tilde{L}^{p,n}_{T_n} + \int_0^T \mathbb{I}_{(T_{n-1},T_n)}(s)\, d\hat{L}^{p,n-1}_s \right).
\end{aligned}$$

Applying Lemma 2.23 at T_n and Lemma 2.22 between T_{n-1} and T_n recursively for $n = N, \ldots, 1$ yields the following theorem.

Theorem 2.24. *Suppose X satisfies the structure condition and K is uniformly bounded. Let $H, H^p \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ be given with Föllmer-Schweizer decompositions*

$$\begin{aligned}
H &= \mathbb{E}(H) + \int_0^T \xi_s \cdot dX_s + L_T, \quad \mathbb{P}\text{-a.s.}, \\
H^p &= \mathbb{E}(H^p) + \int_0^T \xi^p_s \cdot dX_s + L^p_T, \quad \mathbb{P}\text{-a.s.}.
\end{aligned}$$

If we have

$$H^p \xrightarrow[p\to\infty]{L^2} H,$$

then it holds

$$\begin{aligned}
L^p_0 &\xrightarrow[p\to\infty]{L^2} L_0, \\
\xi^p &\xrightarrow[p\to\infty]{L^2(M)} \xi, \\
L^p_T &\xrightarrow[p\to\infty]{L^2} L_T.
\end{aligned}$$

Finally, we can prove the closedness of $\{\mathbb{R} + G_T(\Xi)\}$.

Theorem 2.25. *Suppose X satisfies the structure condition and K is uniformly bounded. Then the subspaces $G_T(\Xi)$ and $\{\mathbb{R} + G_T(\Xi)\}$ are closed subspaces of L^2.*

Proof. Let $(H^p)_{p\in\mathbb{N}_0}$ be a sequence in $G_T(\Xi)$, which satisifies $H^p \xrightarrow[p\to\infty]{L^2} H$. Since for each p, $H^p \in G_T(\Xi)$, there $\exists\, \xi^p \in \Xi$, such that

$$H^p = \int_0^T \xi_s^p \cdot dX_s.$$

This is the unique Föllmer-Schweizer decomposition of H^p. On the other hand, $H \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ admits by Theorem 2.16 a unique Föllmer-Schweizer decomposition

$$H = \mathbb{E}(H) + \int_0^T \xi_s^H \cdot dX_s + L_T^H \quad \mathbb{P}\text{-a.s..}$$

But since $H^p \xrightarrow[p\to\infty]{L^2} H$, Theorem 2.24 yields $\mathbb{E}(H) = 0$, $L_0^H = 0$ $\mathbb{P}$-a.s. and $L_T^H = 0$ $\mathbb{P}$-a.s.. Hence, $H \in G_T(\Xi)$ and $G_T(\Xi)$ is a closed subspace of L^2.
Since $G_T(\Xi)$ is now closed, observe that also $G_T(\Xi)$ plus any finite dimensional subspace of L^2 is closed. Hence, in particular also $\{\mathbb{R} + G_T(\Xi)\}$ is closed. □

In case of X beeing continuous we don't even need the continuity of the Föllmer-Schweizer decomposition to prove the closedness of $G_T(\Xi)$. This can be seen as follows.
Since K is uniformly bounded and X is continuous, we know that

$$Z_T = \mathcal{E}\left(-\int \lambda \cdot dM\right)_T$$

is in $\mathcal{M}^2(\mathbb{P})$, is strictly positive and defines the equivalent minimal martingale measure $\mathbb{Q}_T$ by

$$\frac{d\mathbb{Q}_T}{d\mathbb{P}} := Z_T \quad \text{on } \mathcal{F}_T.$$

Recall the definition $G_t(\xi) = \int_0^t \xi_s \cdot dX_s$. Since $ZG(\xi) \in \mathcal{M}^1_{0,\text{loc}}(\mathbb{P})$, we have $G(\xi) \in \mathcal{M}^1_{0,\text{loc}}(\mathbb{Q}_T)$ and with

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_T}\left(\sup_{0\le t\le T} |G_t(\xi)|\right) &= \mathbb{E}\left(Z_T \sup_{0\le t\le T} |G_t(\xi)|\right) \\
&\le \mathbb{E}\left(Z_T^2\right)^{\frac{1}{2}} \mathbb{E}\left(\left(\sup_{0\le t\le T} |G_t(\xi)|\right)^2\right)^{\frac{1}{2}} \\
&\le \mathbb{E}\left(Z_T^2\right)^{\frac{1}{2}} C \sup_{0\le t\le T} \left\| \int_0^t \xi_s \cdot dM_s \right\|_2 \\
&< \infty,
\end{aligned}$$

we even have $G(\xi) \in \mathcal{M}_0^1(\mathbb{Q}_T)$. Now let $H \in L^2(\mathcal{F}_T, \mathbb{P})$ and $(G_T(\xi^n))_{n\in\mathbb{N}}$ be a sequence in $G_T(\Xi)$, which satisfies

$$G_T(\xi^n) \xrightarrow[n\to\infty]{L^2(\mathbb{P})} H.$$

This implies

$$G_T(\xi^n) \xrightarrow[n\to\infty]{L^1(\mathbb{Q})} H$$

and since $(\mathbb{E}_{\mathbb{Q}_T}(H \mid \mathcal{F}_t))_{0\le t\le T}$ is a martingale with respect to $\mathbb{Q}_T$, we conclude for $t = T$ with the Martingale Representation Theorem that there exists a previsible process ξ, such that

$$H = \int_0^T \xi_s \cdot dX_s.$$

Since X is a continuous local $\mathbb{Q}_T$-martingale, we know $\int \xi \cdot dX$ is a local $\mathbb{Q}_T$-martingale. Together with the Föllmer-Schweizer decomposition of H we obtain

$$\begin{aligned}\int_0^t \xi_s \cdot dX_s &= \mathbb{E}_{\mathbb{Q}_T}(H \mid \mathcal{F}_t)\\ &= \mathbb{E}_{\mathbb{Q}_T}\left(\mathbb{E}(H) + \int_0^T \xi_s^H \cdot dX_s + L_T^H \,\middle|\, \mathcal{F}_t\right)\\ &= \mathbb{E}(H) + \int_0^t \xi_s^H \cdot dX_s + L_t^H \quad \mathbb{P}\text{-a.s.},\end{aligned}$$

since $ZL^H \in \mathcal{M}^1_{\text{loc}}(\mathbb{P})$. For $t = 0$ we conclude $\mathbb{E}(H) + L_0^H = 0$ $\mathbb{P}$-a.s. and taking the previsible quadratic variation yields

$$\left\langle L^H \right\rangle^{\mathbb{Q}_T} = \left\langle Z, L^H \right\rangle^{\mathbb{P}} = 0.$$

Hence, $L_t^H = 0$ $\mathbb{P}$-a.s. for all $0 \le t \le T$, since $\mathbb{E}(L_0^H) = 0$. For $t = 0$ we conclude $\mathbb{E}(H) = 0$ and hence $H \in G_T(\Xi)$.

Corollary 2.26. *Suppose X satisfies the structure condition and K is uniformly bounded. Then the norms $\|\cdot\|_{L^2(M)}$ and $\|G_T(\cdot)\|_2$ are equivalent.*

Proof. For $\xi \in \Xi$ it holds

$$\begin{aligned}\|G_T(\xi)\|_2 &\le \left\| \int_0^T \xi_s \cdot dM_s \right\|_2 + \left\| \int_0^T \xi_s^{tr} dA_s \right\|_2\\ &\le \|\xi\|_{L^2(M)} + \left\| \int_0^T |\xi_s^{tr}\sigma_s\lambda_s| \, dW_s \right\|_2\\ &\le \|\xi\|_{L^2(M)} + \mathbb{E}\left(\int_0^T \xi_s^{tr}\sigma_s\xi_s \, dW_s \int_0^T \lambda_s^{tr}\sigma_s\lambda_s \, dW_s \right)^{1/2}\\ &\le \left(1 + \|K\|_\infty^{1/2}\right) \|\xi\|_{L^2(M)}.\end{aligned}$$

Now assume $G_T(\xi^n) \xrightarrow[n\to\infty]{L^2} G_T(\xi)$. Then with Theorem 2.25 there $\exists\, \xi' \in \Xi$, such that $G_T(\xi^n) \xrightarrow[n\to\infty]{L^2} G_T(\xi')$. But the Föllmer-Schweizer decompositions of H and H^n are given by

$$H = G_T(\xi), \quad H^n = G_T(\xi^n).$$

Continuity of the Föllmer-Schweizer decomposition now yields $\xi^n \xrightarrow[n\to\infty]{L^2(M)} \xi'$. In particular there $\exists\, c > 0$, such that $\|\cdot\|_{L^2(M)} \le c\|G_T(\cdot)\|_2$. □

With Theorem 2.25 we are now able to project $H \in L^2(\mathcal{F}_T, \mathbb{P})$ onto the closed subspaces $G_T(\Xi)$ and $\{\mathbb{R}_+ + G_T(\Xi)\}$. Thus, Hilbert's Projection Theorem immediately proves the following theorem.

Corollary 2.27. *Suppose X satisfies the structure condition and K is uniformly bounded. Then for any $H \in L^2(\mathcal{F}_T, \mathbb{P})$ there exists a unique $\left(V_0^H, \xi^H\right) \in \mathbb{R} \times \Xi$, such that*

$$\mathbb{E}\left(\left(H - V_0^H - \int_0^T \xi_s^H \cdot dX_s\right)^2\right) = \min_{(V_0,\xi)\in\mathbb{R}\times\Xi} \mathbb{E}\left(\left(H - V_0 - \int_0^T \xi_s \cdot dX_s\right)^2\right).$$

Similarly, for any $V_0 \in \mathbb{R}$ there exists a unique $\xi^{V_0} \in \Xi$, such that

$$\mathbb{E}\left(\left(H - V_0 - \int_0^T \xi_s^{V_0} \cdot dX_s\right)^2\right) = \min_{\xi\in\Xi} \mathbb{E}\left(\left(H - V_0 - \int_0^T \xi_s \cdot dX_s\right)^2\right).$$

2.3.2 A special case

The following subsection is based on [Pham et al., 1996].
In practice, we would like to have a more explicit representation of the mean-variance optimal trading strategy ξ^{V_0}. We will prove such a representation in a special case. Let X be continuous, K be uniformly bounded and $\mathbb{Q}_T$ be the minimal martingale measure with density Z_T. Since K is bounded we have $\forall\, 1 \le p < \infty$

$$Z_T \in L^p(\mathcal{F}_T, \mathbb{P}), \quad \frac{1}{Z_T} \in L^p(\mathcal{F}_T, \mathbb{Q}_T). \tag{2.3.2}$$

Hence, Theorem 2.21 yields that Z_T has a general Föllmer-Schweizer decomposition

$$Z_T = \mathbb{E}(Z_T^2) - \mathbb{E}(Z_T L_T^Z) + \int_0^T \zeta_s \cdot dX_s + L_T^Z \quad \mathbb{P}\text{-a.s.}, \tag{2.3.3}$$

where $\zeta \in L^r(M)$ and $L^Z \in \mathcal{M}^r(\mathbb{P})$ with $\mathbb{E}(L_0^Z) = 0$ for every $r < p$ and $p \ge 2$. Starting under $\mathbb{Q}_T$ the constant is obtained by the fact that $\int \zeta \cdot dX$ is $\mathbb{Q}_T$-martingale and finally, by switching back to the $\mathbb{P}$-measure. The goal of this subsection is to prove the following theorem.

Theorem 2.28. *Suppose X is continuous and K is uniformly bounded. Further, assume the special case*

$$L_T^Z = 0 \text{ in Equation (2.3.3).} \tag{2.3.4}$$

Then for fixed $H \in L^{2+\epsilon}(\mathcal{F}_T, \mathbb{P})$, with $\epsilon > 0$, the mean-variance optimal trading strategy ξ^{V_0} is given by

$$\xi_t^{V_0} = \xi_t^{H,\mathbb{Q}_T} - \frac{\zeta_t}{Z_t^{\mathbb{Q}_T}} \left(V_{t-}^{H,\mathbb{Q}_T} - V_0 - \int_0^t \xi_s^{V_0} \cdot dX_s \right), \tag{2.3.5}$$

where

$$\begin{aligned} Z_t^{\mathbb{Q}_T} &:= \mathbb{E}_{\mathbb{Q}_T}(Z_T \,|\, \mathcal{F}_t) = \mathbb{E}(Z_T^2) + \int_0^t \zeta_s \cdot dX_s \quad (0 \le t \le T), \\ V_t^{H,\mathbb{Q}_T} &:= \mathbb{E}_{\mathbb{Q}_T}(H) + \int_0^t \xi_s^{H,\mathbb{Q}_T} \cdot dX_s + L_t^{H,\mathbb{Q}_T} \quad (0 \le t \le T). \end{aligned}$$

Before we prove this theorem it is worth to sketch the idea. By Hilbert's Projection Theorem the mean-variance optimal trading strategy ξ^{V_0} fulfills

$$\mathbb{E}\left(\left(H - V_0 - G_T(\xi^{V_0})\right) G_T(\xi) \right) = \mathbb{E}_{\mathbb{Q}_T}\left(\frac{H - V_0 - G_T(\xi^{V_0})}{Z_T} G_T(\xi) \right) = 0,$$

for every $\xi \in \Xi$. For any $N \in \mathcal{M}^2(\mathbb{Q}_T)$ strongly $\mathbb{Q}_T$-orthogonal to $\mathcal{I}^2(X)$ it holds

$$\mathbb{E}_{\mathbb{Q}_T}(N_T G_T(\xi)) = 0 \quad \forall \, \text{bounded } \xi \in \Xi.$$

Hence, we are looking for an N, such that

$$H - V_0 - G_T(\xi^{V_0}) = N_T Z_T = N_T Z_T^{\mathbb{Q}_T}. \tag{2.3.6}$$

Using the Föllmer-Schweizer decompositions of $H = V_T^{H,\mathbb{Q}_T}$ and $Z_T^{\mathbb{Q}_T}$, together with the product rule applied to $N_T Z_T^{\mathbb{Q}_T}$ we obtain

$$\begin{aligned} &H - V_0 - G_T(\xi^{V_0}) - N_T Z_T^{\mathbb{Q}_T} \\ &\quad = \mathbb{E}_{\mathbb{Q}_T}(H) - V_0 - N_0 \mathbb{E}(Z_T^2) + \int_0^T \left(\xi_s^{H,\mathbb{Q}_T} - \xi_s^{V_0} - N_{s-}\zeta_s \right) \cdot dX_s \\ &\qquad + L_T^{H,\mathbb{Q}_T} - \int_0^T Z_s^{\mathbb{Q}_T} \, dN_s - [N, Z^{\mathbb{Q}_T}]_T. \end{aligned}$$

Since X is continuous and N is strongly $\mathbb{Q}_T$-orthogonal to X^i for each $i = 1, \dots, d$, we have

$$[N, Z^{\mathbb{Q}_T}] = \sum_{i=1}^d \int \zeta^i \, d[N, X^i] = \sum_{i=1}^d \int \zeta^i \, d\left\langle N, X^i \right\rangle^{\mathbb{Q}_T} = 0.$$

Thus, Equation (2.3.6) will hold, if we choose

$$\begin{aligned}
N_t &:= \frac{\mathbb{E}_{\mathbb{Q}_T}(H) - V_0 + L_0^{H,\mathbb{Q}_T}}{\mathbb{E}(Z_T^2)} + \int_0^t \frac{1}{Z_s^{\mathbb{Q}_T}}\, dL_s^{H,\mathbb{Q}_T}, \\
\xi^{V_0} &:= \xi^{H,\mathbb{Q}_T} - N_- \zeta.
\end{aligned}$$

Lemma 2.29. *Define the process N as above. Then for $\epsilon > 0$ we have*

$$N \in \mathcal{M}^{2+\eta}(\mathbb{P}) \quad \text{for } 0 < \eta < \epsilon.$$

Further, N is a $\mathbb{Q}_T$-martingale strongly $\mathbb{Q}_T$-orthogonal to X^i for each $i = 1, \dots, d$ and $N_- \zeta \in L^2(M)$.

Proof. The process $Z_t^{\mathbb{Q}_T}$ is strictly positive and continuous, since X is continuous. Thus, the process N is well defined. By Jensen's Inequality we have

$$\frac{1}{Z_t^{\mathbb{Q}_T}} = \frac{1}{\mathbb{E}_{\mathbb{Q}_T}(Z_T \,|\, \mathcal{F}_t)} \leq \mathbb{E}_{\mathbb{Q}_T}\left(\frac{1}{Z_T} \,\middle|\, \mathcal{F}_t \right)$$

and hence with Doob's Maximal Inequality and (2.3.2)

$$\sup_{0 \leq t \leq T} \frac{1}{Z_T^{\mathbb{Q}_T}} \in L^p(\mathbb{P}) \quad \forall\, 1 < p < \infty.$$

By Theorem 2.21 we have $L^{H,\mathbb{Q}_T} \in \mathcal{M}^{2+\epsilon}(\mathbb{P})$ and therefore with Burkholder-Davis-Gundy's Inequality we obtain for every $\delta < \epsilon/2$

$$\begin{aligned}
[N]_T = \int_0^T \frac{1}{\left(Z_s^{\mathbb{Q}_T}\right)^2}\, d[L^{H,\mathbb{Q}_T}]_s &\leq [L^{H,\mathbb{Q}_T}]_T \sup_{0 \leq t \leq T} \frac{1}{\left(Z_s^{\mathbb{Q}_T}\right)^2} \\
&= [L^{H,\mathbb{Q}_T} - L_0^{H,\mathbb{Q}_T}]_T \sup_{0 \leq t \leq T} \frac{1}{\left(Z_s^{\mathbb{Q}_T}\right)^2} \in L^{1+\delta}(\mathbb{P}).
\end{aligned}$$

This proves $N \in \mathcal{M}^{2+\eta}(\mathbb{P})$.
Note, that $L^{H,\mathbb{Q}_T}$ is strongly $\mathbb{P}$-orthogonal to each M^i for $i = 1, \dots, d$. Hence with Theorem 2.18 and the definition of N, we conclude that N is a $\mathbb{Q}_T$-martingale strongly $\mathbb{Q}_T$-orthogonal to X^i for each i. Finally, since $\zeta \in L^r(M)$ for every $r < p$, it holds for every $\delta < \epsilon/2$

$$\int_0^T N_{s-} \zeta_s^{tr} \sigma_s \zeta_s N_{s-}\, dW_s \leq \left(\sup_{0 \leq s \leq T} |N_s|^2 \right) \int_0^T \zeta_s^{tr} \sigma_s \zeta_s\, dW_s \in L^{1+\delta}(\mathbb{P}).$$

This shows $N_- \zeta \in L^2(M)$. $\square$

Proof of Theorem 2.28. We first prove that $\xi^{V_0} := \xi^{H,\mathbb{Q}_T} - N_- \zeta$ is equal to (2.3.5). From Lemma 2.29 we know that N is a $\mathbb{Q}_T$-martingale strongly $\mathbb{Q}_T$-orthogonal to X^i for each $i = 1, \dots, d$. Since X is continuous and $L^Z_T = 0$, we obtain

$$[N, Z^{\mathbb{Q}_T}] = \sum_{i=1}^d \int \zeta^i \, d[N, X^i] = \sum_{i=1}^d \int \zeta^i \, d \left\langle N, X^i \right\rangle^{\mathbb{Q}_T} = 0$$

and hence by the product rule and the respective definitions

$$\begin{aligned} NZ^{\mathbb{Q}_T} &= N_0 \mathbb{E}(Z_T^2) + \int N_- \zeta \cdot dX + \int Z^{\mathbb{Q}_T} \, dN \\ &= \mathbb{E}_{\mathbb{Q}_T}(H) - V_0 + \int \left(\xi^{H,\mathbb{Q}_T} - \xi^{V_0} \right) \cdot dX + L^{H,\mathbb{Q}_T} \\ &= V^{H,\mathbb{Q}_T} - V_0 - \int \xi^{V_0} \cdot dX. \end{aligned}$$

With the continuity of $Z^{\mathbb{Q}_T}$ we conclude that

$$\xi^{V_0} = \xi^{H,\mathbb{Q}_T} - \frac{\zeta}{Z^{\mathbb{Q}_T}} N_- Z_-^{\mathbb{Q}_T}$$

equals (2.3.5).
Now let us prove that ξ^{V_0} is mean-variance optimal. By construction we have

$$H - V_0 - G_T(\xi^{V_0}) = N_T Z_T^{\mathbb{Q}_T} = N_T Z_T.$$

By the strong $\mathbb{Q}_T$-orthogonality of N and X^i for each i, we have that $NG(\xi)$ is a local $\mathbb{Q}_T$-martingale with $N_0 G_0(\xi) = 0$ for every $\xi \in \Xi$. With Lemma 2.29 and Hölder's Inequality we have for every $\delta < \epsilon/2$

$$\sup_{0 \le t \le T} |N_t G_t(\xi)| \in L^{1+\delta}(\mathbb{P}),$$

which implies with (2.3.2)

$$\sup_{0 \le t \le T} |N_t G_t(\xi)| \in L^1(\mathbb{Q}_T),$$

Thus, $NG(\xi)$ is even a true $\mathbb{Q}_T$-martingale and consequently

$$\mathbb{E}\left(\left(H - V_0 - G_T(\xi^{V_0}) \right) G_T(\xi) \right) = \mathbb{E}_{\mathbb{Q}_T}(N_T G_T(\xi)) = 0 \quad \text{for every } \xi \in \Xi$$

and by Hilbert's Projection Theorem is ξ^{V_0} mean-variance optimal. $\square$

Corollary 2.30. *Suppose we have the same assumptions as in Theorem 2.28. Then the minimal residual risk* $J_0 := \mathbb{E}\left(\left(H - V_0 - G_T(\xi^{V_0}) \right)^2 \right)$ *is*

$$J_0 = \frac{(\mathbb{E}_{\mathbb{Q}_T}(H) - V_0)^2 + \mathbb{E}\left(\left(L_0^{H,\mathbb{Q}_T} \right)^2 \right)}{\mathbb{E}(Z_T^2)} + \mathbb{E}_{\mathbb{Q}_T}\left(\int_0^T \frac{1}{Z_s^{\mathbb{Q}_T}} \, d[L^{H,\mathbb{Q}_T}]_s \right).$$

Proof. With $H - V_0 - G_T(\xi^{V_0}) = N_T Z_T$ and $\mathbb{E}_{\mathbb{Q}_T}(N_T G_T(\xi)) = 0$ for every $\xi \in \Xi$ we obtain

$$\begin{aligned}\mathbb{E}\left(\left(H - V_0 - G_T(\xi^{V_0})\right)^2\right) &= \mathbb{E}_{\mathbb{Q}_T}\left(N_T\left(H - V_0 - G_T(\xi^{V_0})\right)\right)\\ &= \mathbb{E}_{\mathbb{Q}_T}\left(N_T\left(\mathbb{E}_{\mathbb{Q}_T}(H) - V_0 + G_T(\xi^{H,\mathbb{Q}_T} - \xi^{V_0}) + L_T^{H,\mathbb{Q}_T}\right)\right)\\ &= \mathbb{E}_{\mathbb{Q}_T}\left(N_T(\mathbb{E}_{\mathbb{Q}_T}(H) - V_0)\right) + \mathbb{E}_{\mathbb{Q}_T}(N_T L_T^{H,\mathbb{Q}_T}).\end{aligned}$$

From Lemma 2.29 we know that N is a $\mathbb{Q}_T$-martingale and we conclude

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}_T}\left(N_T(\mathbb{E}_{\mathbb{Q}_T}(H) - V_0)\right) &= \left(\mathbb{E}_{\mathbb{Q}_T}(H) - V_0\right)\mathbb{E}_{\mathbb{Q}_T}(N_0)\\ &= \frac{(\mathbb{E}_{\mathbb{Q}_T}(H) - V_0)^2 + (\mathbb{E}_{\mathbb{Q}_T}(H) - V_0)\,\mathbb{E}_{\mathbb{Q}_T}(L_0^{H,\mathbb{Q}_T})}{\mathbb{E}(Z_T^2)}\\ &= \frac{(\mathbb{E}_{\mathbb{Q}_T}(H) - V_0)^2}{\mathbb{E}(Z_T^2)},\end{aligned}$$

since $\mathbb{E}_{\mathbb{Q}_T}(L_0^{H,\mathbb{Q}_T}) = 0$. For the second part, since $L^{H,\mathbb{Q}_T}, N \in \mathcal{M}^2(\mathbb{Q}_T)$, we get

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}_T}(N_T L_T^{H,\mathbb{Q}_T}) &= \mathbb{E}_{\mathbb{Q}_T}(N_0 L_0^{H,\mathbb{Q}_T}) + \mathbb{E}_{\mathbb{Q}_T}\left([N, L^{H,\mathbb{Q}_T}]_T\right)\\ &= \frac{\mathbb{E}_{\mathbb{Q}_T}\left((L_0^{H,\mathbb{Q}_T})^2\right)}{\mathbb{E}(Z_T^2)} + \mathbb{E}_{\mathbb{Q}_T}\left(\int_0^T \frac{1}{Z_s^{\mathbb{Q}_T}}\, d[L^{H,\mathbb{Q}_T}]_s\right).\end{aligned}$$

The observation $\mathbb{Q}_T = \mathbb{P}$ on $\mathcal{F}_0$ concludes the proof. □

Remark. Note, that the mean-variance optimal pair $\left(V_0^H, \xi^H\right) \in \mathbb{R} \times \Xi$ of Definition 2.13 is given by

$$\left(V_0^H, \xi^H\right) := \left(\mathbb{E}_{\mathbb{Q}_T}(H), \xi^{V_0}\right).$$

The obtained results are only possible, because we assumed $L_T^Z = 0$. The question arises for which class of examples our special assumption is satisfied. We will show this in case where the terminal value K_T of the mean-variance tradeoff process is **deterministic**.

Suppose X is continuous and Z_T defines the minimal martingale measure. Then continuity of K and the product rule for semimartingales imply

$$Z = \mathcal{E}\left(-\int \lambda \cdot dM\right) = \mathcal{E}\left(-\int \lambda \cdot dX + K\right) = \mathcal{E}\left(-\int \lambda \cdot dX\right) e^K,$$

since $\left\langle -\int \lambda \cdot dX, K\right\rangle = 0$. Thus, we obtain for the terminal value

$$\begin{aligned}Z_T = e^{K_T}\mathcal{E}\left(-\int \lambda \cdot dX\right)_T &= e^{K_T}\left(1 - \int_0^T \mathcal{E}\left(-\int \lambda \cdot dX\right)_s \lambda_s \cdot dX_s\right)\\ &= e^{K_T} + \int_0^T \underbrace{-e^{K_T}\mathcal{E}\left(-\int \lambda \cdot dX\right)_s \lambda_s}_{=:\zeta_s} \cdot dX_s.\end{aligned}$$

Under the assumption of K_T beeing deterministic we conclude that Z_T can be written as the sum of a constant and a stochastic integral, where $\zeta \in \Xi$ since K is obviously uniformly bounded. Hence, the special assumption (2.3.4) is fulfilled. Since $\mathcal{E}\left(-\int \lambda \cdot dX\right)$ is a $\mathbb{Q}_T$-martingale, we further have

$$Z_t^{\mathbb{Q}_T} = \mathbb{E}_{\mathbb{Q}_T}(Z_T \,|\, \mathcal{F}_t) = e^{K_T}\mathcal{E}\left(-\int \lambda \cdot dX\right)_t \quad (0 \le t \le T),$$

and hence

$$-\frac{\zeta_t}{Z_t^{\mathbb{Q}_T}} = \lambda_t \quad (0 \le t \le T),$$

which simplifies the mean-variance optimal trading strategy ξ^{V_0} of Theorem 2.28 to

$$\xi_t^{V_0} = \xi_t^{H,\mathbb{Q}_T} + \lambda_t\left(V_{t-}^{H,\mathbb{Q}_T} - V_0 - \int_0^t \xi_s^{V_0} \cdot dX_s\right) \quad (0 \le t \le T). \tag{2.3.7}$$

Using the obtained representations of $Z^{\mathbb{Q}_T}$ and Z, as well as the fact that $1/Z$ is the density process of $\mathbb{P}$ with respect to $\mathbb{Q}_T$ we have

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}_T}\left(\int_0^T \frac{1}{Z_s^{\mathbb{Q}_T}} d[L^{H,\mathbb{Q}_T}]_s\right) &= e^{-K_T}\,\mathbb{E}_{\mathbb{Q}_T}\left(\int_0^T \frac{1}{\mathcal{E}\left(-\int \lambda \cdot dX\right)_s} \frac{e^{K_s}}{e^{K_s}}\, d[L^{H,\mathbb{Q}_T}]_s\right)\\ &= e^{-K_T}\,\mathbb{E}\left(\int_0^T e^{K_s}\, d[L^{H,\mathbb{Q}_T}]_s\right).\end{aligned}$$

Thus, the remaining quadratic risk of Corollary 2.30 is given by

$$J_0 = e^{-K_T}\left(\left(\mathbb{E}_{\mathbb{Q}_T}(H) - V_0\right)^2 + \mathbb{E}\left((L_0^{H,\mathbb{Q}_T})^2\right) + \mathbb{E}\left(\int_0^T e^{K_s}\, d[L^{H,\mathbb{Q}_T}]_s\right)\right). \tag{2.3.8}$$

If one wants to find a general explicit representation of the mean-variance optimal trading strategy, one has to work with the **variance optimal martingale measure** $\mathbb{Q}_T^{opt}$. It is defined by its density $d\mathbb{Q}_T^{opt}/d\mathbb{P}$, which minimizes the $L^2(\mathbb{P})$-norm. In [Delbaen and Schachermayer, 1996], Lemma 2.2, it is then proven that $d\mathbb{Q}_T^{opt}/d\mathbb{P}$ always satisfies our special assumption (2.3.4). In particular it is proven, that the assumption (2.3.4) is equivalent to the assumption that the minimal martingale measure $\mathbb{Q}_T$ and the variance optimal martingale measure $\mathbb{Q}_T^{opt}$ coincide. For more details in this direction we refer to [Delbaen et al., 1997] and for an overview to section four of [Schweizer, 1999].

As a final observation, note that in case of a complete market the minimal and the variance optimal martingale measure coincide, since there is obviously only one equivalent martingale measure. Hence, the special assumption (2.3.4) is automatically satisfied[13].

[13] This can also be seen by the martingale representation theorem, as in most cases a d-dimensional Brownian motion B and its $\mathbb{P}$-augmented filtration $\mathcal{F}^B$ are used in the description of X.

3 Application to life insurance

This chapter is based on [Møller, 1998]. Detailed background information can be found in [Møller and Steffensen, 2007].
We would like to apply the introduced theory of quadratic hedging to a portfolio of unit-linked life insurance contracts. In particular, we consider only *pure endowment* insurance policies, since these contracts can be directly casted into our setting of European type contingent claims.

3.1 The model

The financial market

We start by introducing the financial market. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space equipped with the $\mathbb{P}$-augmented filtration $\mathcal{G} = (\mathcal{G}_t)_{0 \leq t \leq T}$ of a standard one dimensional Brownian motion[1] $B = (B_t)_{0 \leq t \leq T}$ on a finite time horizon $T > 0$. Let the market consist of two tradeable assets with real-valued price processes $\tilde{S}^i = (\tilde{S}^i_t)_{0 \leq t \leq T}$, for $i = 0, 1$. $\tilde{S}^0 > 0$ represents the riskless bond or bank account and $\tilde{S}^1$ the risky asset. Their dynamics are given by[2]

$$d\tilde{S}^1_t = \tilde{S}^1_t \left(m(t, \tilde{S}^1_t)\, dt + \sigma(t, \tilde{S}^1_t)\, dB_t \right), \quad \tilde{S}^1_0 > 0, \tag{3.1.1}$$

$$d\tilde{S}^0_t = \tilde{S}^0_t r(t, \tilde{S}^1_t)\, dt, \quad \tilde{S}^0_0 = 1. \tag{3.1.2}$$

Here $\tilde{S}^1_0$ is not random. m can be interpreted as the mean return rate of $\tilde{S}^1$, σ as the volatility and r as the interest short rate. A solution to Equation (3.1.1) exists for smooth enough functions $xm(t, x)$ and $x\sigma(t, x)$, namely Lipschitz continuity in x and admission of a linear growth condition. Both regularity assumptions we shall assume henceforth. Furthermore, assume $r(t, x)$ is continuous and satisfies

$$\mathbb{E}\left(\exp\left(\int_0^T r(s, \tilde{S}^1_s)\, ds \right) \right) < \infty.$$

[1] We assume that *all* paths of B are continuous. In case B' is only a Brownian motion with $\mathbb{P}$-a.s. continuous paths, let N denote the exceptional set and define $B = B'$ on N^c and $B = 0$ on N.
[2] For constant m, σ and r this is the Back-Scholes Model.

For $x\sigma(t,x)$ we assume in addition $\forall (t,x) \in [0,T] \times \mathbb{R}_+ \, \exists \, \epsilon > 0$, such that $x\sigma(t,x) > \epsilon$. We need this to ensure the well definedness of the *market price of risk*

$$\lambda_t := \frac{m(t, \tilde{S}^1_t) - r(t, \tilde{S}^1_t)}{\sigma(t, \tilde{S}^1_t)} \quad (0 \leq t \leq T),$$

which we assume to be uniformly bounded. Then, we can define a change of measure density process $Z = (Z_t)_{0 \leq t \leq T}$ by

$$Z_t := \mathcal{E}\left(-\int_0^t \lambda_s \, dB_s\right) = \exp\left(-\int_0^t \lambda_s \, dB_s - \frac{1}{2} \underbrace{\int_0^t \lambda_s^2 \, ds}_{=K_t}\right) \quad (0 \leq t \leq T).$$

Observe, that the mean-variance tradeoff proccess K of Definition 2.9 is **uniformly bounded** since the market price of risk is assumed to be uniformly bounded. Further, Z is a local $\mathbb{P}$-martingale and since Novikov's Condition

$$\mathbb{E}\left(\exp\left(\frac{1}{2} K_T\right)\right) < \infty$$

is fulfilled, we even have that Z is a true $\mathbb{P}$-martingale. Thus, we can apply Girsanov's Theorem and

$$\frac{d\mathbb{Q}_T}{d\mathbb{P}} = Z_T \quad \text{on } \mathcal{G}_T$$

defines a probability measure $\mathbb{Q}_T$ equivalent to $\mathbb{P}$, such that the process

$$B_t^{\mathbb{Q}_T} := B_t + \int_0^{t \wedge T} \lambda_s \, ds \quad t \geq 0,$$

is a $\mathbb{Q}_T$-standard Brownian motion. Applying the product rule to the **continuous** process $X_t = S^1_t = \tilde{S}^1_t / \tilde{S}^0_t$ now yields with $(\tilde{S}^0_t)^{-1} = \exp(-\int_0^t r(s, \tilde{S}^1_s) \, ds)$

$$\begin{aligned}
dX_t &= d\left(e^{-\int_0^t r(s,\tilde{S}^1_s)\, ds} \tilde{S}^1_t\right) \\
&= d\left(e^{-\int_0^t r(s,\tilde{S}^1_s)\, ds}\right) \tilde{S}^1_t + e^{-\int_0^t r(s,\tilde{S}^1_s)\, ds} d\tilde{S}^1_t + d\left[e^{-\int_0^t r(s,\tilde{S}^1_s)\, ds}, \tilde{S}^1_t\right] \\
&= -r(t, \tilde{S}^1_t) e^{-\int_0^t r(s,\tilde{S}^1_s)\, ds} \tilde{S}^1_t \, dt + e^{-\int_0^t r(s,\tilde{S}^1_s)\, ds} \tilde{S}^1_t \left(m(t, \tilde{S}^1_t) \, dt + \sigma(t, \tilde{S}^1_t) \, dB_t\right) \\
&= X_t \left(\left(m(t, \tilde{S}^1_t) - r(t, \tilde{S}^1_t)\right) dt + \sigma(t, \tilde{S}^1_t) \, dB_t\right) \\
&= \sigma(t, \tilde{S}^1_t) X_t \, dB_t^{\mathbb{Q}_T}.
\end{aligned}$$

The explicit solution is[3]

$$
\begin{aligned}
X_t &= X_0 \exp\left(\int_0^t \left(m(s, \tilde{S}_s^1) - r(s, \tilde{S}_s^1) - \frac{1}{2}\sigma^2(s, \tilde{S}_s^1) \right) ds + \int_0^t \sigma(s, \tilde{S}_s^1)\, dB_s \right) \\
&= X_0 \exp\left(\int_0^t \sigma(s, \tilde{S}_s^1)\, dB_s^{\mathbb{Q}_T} - \frac{1}{2} \int_0^t \sigma^2(s, \tilde{S}_s^1)\, ds \right) \quad (0 \le t \le T).
\end{aligned}
$$

Thus, X is a local $\mathbb{Q}_T$-martingale and by Theorem 1.2 the financial market model is arbitrage free. Since $\mathbb{Q}_T$ is unique on $\mathcal{G}_T$, the financial market model is even complete. Suppose now we have an *undiscounted* square-integrable claim $\tilde{H}$. Its price process $\tilde{V} = (\tilde{V}_t)_{0 \le t \le T}$ is then given by

$$
\tilde{V}_t = \mathbb{E}_{\mathbb{Q}_T}\left(\frac{\tilde{S}_t^0}{\tilde{S}_T^0} \tilde{H} \,\middle|\, \mathcal{G}_t \right)
$$

and $\tilde{H}$ is attainable by a self-financing, previsible strategy ϕ (compare the Definitions 1.1 and 1.5). Assume further, that for some $\mathbb{Q}_T$-integrable function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ we have $\tilde{H} = f(\tilde{S}_T^1)$. Then the price process can be written as $\tilde{V}_t = F(t, \tilde{S}_t^1)$ with

$$
F(t, s) = \mathbb{E}_{\mathbb{Q}_T}\left(\exp\left(-\int_t^T r(u, \tilde{S}_u^1)\, du \right) f(\tilde{S}_T^1) \,\middle|\, \tilde{S}_t^1 = s \right).
$$

Assuming that all regularity conditions of the **Feyman Kac Formula** are fulfilled, we can now link F to the smooth enough solution of the partial differential equation

$$
\begin{aligned}
u_t(t, s) + r(t, s)s\, u_s(t, s) + \frac{1}{2}\sigma^2(t, s)s^2 u_{ss}(t, s) - r(t, s)u(t, s) &= 0 \quad (t, s) \in [0, T] \times \mathbb{R}_+, \\
u(T, s) &= f(s) \quad s \in \mathbb{R}_+, \\
u(t, 0) &= 0 \quad t \in [0, T],
\end{aligned}
\tag{3.1.3}
$$

where the subscripts denote the respective partial derivatives. For the conditions on the existence of such a smooth enough solution we refer the reader to the general PDE literature.

The insurance portfolio

Now let us turn to the description of the insurance portfolio. Suppose we have a portfolio of x-year old individuals and each has the same unit-linked pure endowment contract with survival benefit $f(\tilde{S}_T^1) \in L^2(\mathbb{P})$ for some continuous function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$. Let $l_x \in \mathbb{N}$ denote the total number of persons we have in our portfolio and

[3] Can be validated with Itô's formula.

$(T_i)_{i=1,\ldots,l_x}$ denote the remaining life lengths of the insured. We assume that the non-negative random variables $(T_i)_{i=1,\ldots,l_x}$ are i.i.d. on $(\Omega, \mathcal{A}, \mathbb{P})$ with continuous *force of mortality* μ_{x+t}. Thus, the individual survival function is given by

$$\begin{aligned}
{}_tp_x &= \mathbb{P}(T_i > t) = \exp\left(-\int_0^t \mu_{x+s}\,ds\right), \\
\frac{\partial {}_tp_x}{\partial t} &= -\mu_{x+t}\,{}_tp_x.
\end{aligned}$$

Now, define the counting process $N = (N_t)_{0\le t\le T}$ by

$$N_t := \sum_{i=1}^{l_x} \mathbb{I}_{\{T_i \le t\}},$$

which counts the dead in our portfolio and let $\mathcal{H} = (\mathcal{H}_t)_{0\le t\le T}$ with $\mathcal{H}_t = \sigma\{N_s \mid s \le t\}$ be the generating filtration of N. By construction, N is càdlàg. Furthermore, N is a Markov process under $\mathcal{H}$. Observing, that

$$\mathbb{E}(N_t - N_{t-} \mid \mathcal{H}_{t-}) = (l_x - N_{t-})\mu_{x+t}\,dt$$

should coincide with '$\rho_t\,dt$', we can define the non-negative, previsible intensity $\rho = (\rho_t)_{0\le t\le T}$ with $\int_0^T \rho_s\,ds < \infty$ $\mathbb{P}$-a.s. by

$$\rho_t = (l_x - N_{t-})\mu_{x+t} \quad (0 \le t \le T).$$

The compensated counting process $M = (M_t)_{0\le t\le T}$ is then given by

$$M_t = N_t - \int_0^t \rho_s\,ds \quad (0 \le t \le T),$$

which is a $\mathbb{P}$-martingale under $\mathcal{H}$.

The combined model

In the combined model we assume independence between $\mathcal{G}$ and $\mathcal{H}$ under $\mathbb{P}$ and use the filtration $\mathcal{F} = (F_t)_{0\le t\le T}$ given by

$$\mathcal{F}_t = \mathcal{H}_t \vee \mathcal{G}_t \quad (0 \le t \le T)$$

as information flow. Here, $\mathcal{F}_0$ is $\mathbb{P}$-trivial. At time $t = 0$ each of the l_x individuals signs an endowment contract with discounted benefit

$$H_i = \mathbb{I}_{\{T_i > T\}} \frac{1}{\tilde{S}_T^0} f(\tilde{S}_T^1)$$

at maturity T. The total portfolio claim at maturity is then given by

$$H = (l_x - N_T)\frac{1}{\tilde{S}_T^0} f(\tilde{S}_T^1).$$

The insurance company is now allowed to trade freely without transaction costs, taxes or short sale restrictions any unit of the underlying assets $\tilde{S}_t^0$ and $\tilde{S}_t^1$ on the time horizon $[0, T]$. Note, that the combined model with respect to $\mathcal{F}$ is incomplete. Intuitively this is clear, since we have the additional death uncertainty of each individual.
In detail, this can be seen as follows.

Suppose $h = (h_t)_{0\le t\le T}$ is an $\mathcal{H}$-previsible, non-negative process with $\int_0^T h_s\rho_s\,ds < \infty$ $\mathbb{P}$-a.s.. Define $U = (U_t)_{0\le t\le T}$ by

$$\begin{aligned} dU_t &= dU_{t-}(h_t - 1)dM_t, \\ U_0 &= 1. \end{aligned}$$

Under the assumption $\mathbb{E}(U_T) = 1$, we know by construction that U is an $\mathcal{H}$-martingale and we can define a new probability measure

$$\begin{aligned} \frac{d\mathbb{P}^*}{d\mathbb{P}} &= Z_T U_T \quad \text{on } \mathcal{F}_T, \\ \frac{d\mathbb{P}^*}{d\mathbb{P}} &= Z_t U_t \quad \text{on } \mathcal{F}_t. \end{aligned}$$

We obtain that $X_t = S_t^1 = \tilde{S}_t^1/\tilde{S}_t^0$ is also a $\mathbb{P}^*$-martingale, since for $s \le t$

$$\mathbb{E}_{\mathbb{P}^*}(X_t \,|\, \mathcal{F}_s) = \frac{\mathbb{E}(X_t Z_T U_T \,|\, \mathcal{F}_s)}{\mathbb{E}(Z_T U_T \,|\, \mathcal{F}_s)} = \frac{\mathbb{E}(X_t Z_T \,|\, \mathcal{G}_s)}{\mathbb{E}(Z_T \,|\, \mathcal{G}_s)}\frac{\mathbb{E}(U_T \,|\, \mathcal{H}_s)}{\mathbb{E}(U_T \,|\, \mathcal{H}_s)} = \mathbb{E}_{\mathbb{Q}_T}(X_t \,|\, \mathcal{G}_s) = X_s,$$

where we used the independence of $\mathcal{G}$ and $\mathcal{H}$ under $\mathbb{P}$. Hence, for any h with $\mathbb{E}(U_T) = 1$ we have an equivalent martingale measure on $\mathcal{F}_T$. Consequently, the equivalent martingale measure is not unique on $\mathcal{F}_T$ and with Theorem 1.3 we conclude that the combined market is incomplete.

The question arises, which equivalent martingale measure to use. To obtain the **minimal martingale measure** $\mathbb{P}^* = \mathbb{Q}_T$, we choose $h = 1$[4]. Note, that this choice is in perfect analogy with the usual assumption of *risk neutrality with respect to mortality.* Finally, observe that $M_t = N_t - \int_0^t \rho_s\,ds$ is a $\mathbb{Q}_T$-martingale under $\mathcal{F}$, since the change of measure only affects the financial part.

3.2 Pure endowment policies

From Theorem 2.20 we know we need to find the decomposition of

$$V_t^{H,\mathbb{Q}_T} = \mathbb{E}_{\mathbb{Q}_T}(H \,|\, \mathcal{F}_t) \quad (0 \le t \le T).$$

[4] Compare Theorem 2.17 and the proof of Theorem 2.18 (i).

Since N and $(\tilde{S}^0, \tilde{S}^1)$ are stochastically independent under $\mathbb{Q}_T$ we get

$$V_t^{H,\mathbb{Q}_T} = \mathbb{E}_{\mathbb{Q}_T}\left(l_x - N_T \,|\, \mathcal{H}_t\right) \frac{1}{\tilde{S}_t^0} \mathbb{E}_{\mathbb{Q}_T}\left(\frac{\tilde{S}_t^0}{\tilde{S}_T^0} f(\tilde{S}_T^1) \,\middle|\, \mathcal{G}_t \right).$$

For the insurance part we get

$$\mathbb{E}_{\mathbb{Q}_T}\left(l_x - N_T \,|\, \mathcal{H}_t\right) = \mathbb{E}_{\mathbb{Q}_T}\left(\sum_{i=1}^{l_x} \mathbb{I}_{\{T_i > T\}} \,\middle|\, \mathcal{H}_t \right) = (l_x - N_t)_{T-t}p_{x+t}$$

and from the previous section we know

$$F(t, \tilde{S}_t^1) = \mathbb{E}_{\mathbb{Q}_T}\left(\frac{\tilde{S}_t^0}{\tilde{S}_T^0} f(\tilde{S}_T^1) \,\middle|\, \mathcal{G}_t \right)$$

is obtained by solving the PDE (3.1.3). Hence, we have

$$\begin{aligned} V_t^{H,\mathbb{Q}_T} &= (l_x - N_t)_{T-t}p_{x+t} \frac{1}{\tilde{S}_t^0} F(t, \tilde{S}_t^1) \quad (0 \le t \le T), \\ V_0^{H,\mathbb{Q}_T} &= l_x \, {}_Tp_x F(0, \tilde{S}_0^1). \end{aligned}$$

Since $\mathcal{F}_0$ is $\mathbb{P}$-trivial, we try to find the Galtchouk-Kunita-Watanabe decomposition

$$V_t^{H,\mathbb{Q}_T} = V_0^{H,\mathbb{Q}_T} + \int_0^t \xi_s^{H,\mathbb{Q}_T}\, dX_s + L_t^{H,\mathbb{Q}_T} \quad \mathbb{Q}_T\text{-a.s. (and } \mathbb{P}\text{-a.s.)},$$

where $V_0^{H,\mathbb{Q}_T} \in \mathbb{R}_+$, $\xi^{H,\mathbb{Q}_T}$ is a strategy in the sense of Definition 2.1 with respect to $\mathbb{Q}_T$ and $L^{H,\mathbb{Q}_T} \in \mathcal{M}_0^2(\mathbb{Q}_T)$, which is strongly orthogonal to $\mathcal{I}^2(X)$. Observing that only jumps of $(l_x - N_t)$ cause discontinuities in $V^{H,\mathbb{Q}_T}$, we obtain with Itô's Formula

$$\begin{aligned} V_t^{H,\mathbb{Q}_T} = V_0^{H,\mathbb{Q}_T} &+ \int_0^t (l_x - N_{u-}) \frac{1}{\tilde{S}_u^0} F(u, \tilde{S}_u^1)_{T-u}p_{x+u}\mu_{x+u}\, du \\ &+ \int_0^t (l_x - N_{u-})_{T-u}p_{x+u}\, d\left(\frac{1}{\tilde{S}_u^0} F(u, \tilde{S}_u^1) \right) + \sum_{0<u\le t} \left(V_u^{H,\mathbb{Q}_T} - V_{u-}^{H,\mathbb{Q}_T} \right). \end{aligned} \tag{3.2.1}$$

Using the PDE (3.1.3) we calculate

$$\begin{aligned}
d\left(\frac{1}{\tilde S^0_t}F(t,\tilde S^1_t)\right) &= -r(t,\tilde S^1_t)\frac{1}{\tilde S^0_t}F(t,\tilde S^1_t)\,dt \\
&\quad + \frac{1}{\tilde S^0_t}\left(\left(F_t(t,\tilde S^1_t)+\frac{1}{2}F_{ss}(t,\tilde S^1_t)\sigma^2(t,\tilde S^1_t)(\tilde S^1_t)^2\right)dt + F_s(t,\tilde S^1_t)\,d\tilde S^1_t\right) \\
&= \frac{1}{\tilde S^0_t}\left(F_t(t,\tilde S^1_t)+\frac{1}{2}F_{ss}(t,\tilde S^1_t)\sigma^2(t,\tilde S^1_t)(\tilde S^1_t)^2 - r(t,\tilde S^1_t)\frac{1}{\tilde S^0_t}F(t,\tilde S^1_t)\right)dt \\
&\quad + \frac{1}{\tilde S^0_t}F_s(t,\tilde S^1_t)\,d\tilde S^1_t \\
&= \frac{1}{\tilde S^0_t}\left(-r(t,\tilde S^1_t)\tilde S^1_t F_s(t,\tilde S^1_t)\right)dt + \frac{1}{\tilde S^0_t}F_s(t,\tilde S^1_t)\,d\tilde S^1_t \\
&= -r(t,\tilde S^1_t)X_tF_s(t,\tilde S^1_t)\,dt + F_s(t,\tilde S^1_t)X_t\left(m(t,\tilde S^1_t)\,dt+\sigma(t,\tilde S^1_t)\,dB_t\right) \\
&= F_s(t,\tilde S^1_t)X_t\left(\left(m(t,\tilde S^1_t)-r(t,\tilde S^1_t)\right)dt+\sigma(t,\tilde S^1_t)\,dB_t\right) \\
&= F_s(t,\tilde S^1_t)\,dX_t.
\end{aligned}$$

The discontinuities can be expressed as

$$\begin{aligned}
\sum_{0<u\le t}\left(V^{H,\mathbb{Q}_T}_u - V^{H,\mathbb{Q}_T}_{u-}\right) &= \sum_{0<u\le t}(-N_u+N_{u-})_{T-u}p_{x+u}\frac{1}{\tilde S^0_u}F(u,\tilde S^1_u) \\
&= -\int_0^t {}_{T-u}p_{x+u}\frac{1}{\tilde S^0_u}F(u,\tilde S^1_u)\,dN_u.
\end{aligned}$$

Inserting both results in (3.2.1) and recalling that $M_t = N_t - \int_0^t \rho_u\,du$ with $\rho_u = (l_x - N_{u-})\mu_{x+u}$ is a $\mathbb{Q}_T$-martingale under $\mathcal{F}$ yields

$$\begin{aligned}
V^{H,\mathbb{Q}_T}_t &= V^{H,\mathbb{Q}_T}_0 + \int_0^t (l_x-N_{u-})\frac{1}{\tilde S^0_u}F(u,\tilde S^1_u)_{T-u}p_{x+u}\mu_{x+u}\,du \\
&\quad + \int_0^t (l_x-N_{u-})_{T-u}p_{x+u}F_s(u,\tilde S^1_u)\,dX_u - \int_0^t {}_{T-u}p_{x+u}\frac{1}{\tilde S^0_u}F(u,\tilde S^1_u)\,dN_u \\
&= V^{H,\mathbb{Q}_T}_0 + \int_0^t \underbrace{(l_x-N_{u-})_{T-u}p_{x+u}F_s(u,\tilde S^1_u)}_{=:\,\xi^{H,\mathbb{Q}_T}_u}\,dX_u + \underbrace{\int_0^t -\frac{1}{\tilde S^0_u}F(u,\tilde S^1_u)_{T-u}p_{x+u}\,dM_u}_{=:\,L^{H,\mathbb{Q}_T}_t}
\end{aligned}$$

and hence we have found the Galtchouk-Kunita-Watanabe decomposition of $V^{H,\mathbb{Q}_T}_t$ under $\mathbb{Q}_T$. Note, that $L^{H,\mathbb{Q}_T}$ is strongly orthogonal to $\mathcal{I}^2(X)$ since pure jump and continuous martingales are orthogonal. Using Theorem 2.20 and Theorem 2.12 now leads to the following corollary.

Corollary 3.1. *The pseudo-optimal strategy* $\phi^H = \big((\phi_t^0)^H, \xi_t^H\big)$ *for the claim* $H = (l_x - N_T)(\tilde{S}_T^0)^{-1} f(\tilde{S}_T^1)$ *is given by*

$$\xi_t^H = \xi_t^{H,\mathbb{Q}_T} = (l_x - N_{t-})_{T-t}p_{x+t} F_s(t, \tilde{S}_t^1) \quad (0 \le t \le T),$$

$$(\phi_t^0)^H = V_t^{H,\mathbb{Q}_T} - \xi_t^{H,\mathbb{Q}_T} X_t = (l_x - N_t)_{T-t}p_{x+t} \frac{1}{\tilde{S}_t^0} F(t, \tilde{S}_t^1) - \xi_t^{H,\mathbb{Q}_T} X_t \quad (0 \le t \le T),$$

with minimal remaining risk

$$\mathbb{E}\left(\left(L_T^{H,\mathbb{Q}_T} - L_t^{H,\mathbb{Q}_T}\right)^2 \,\middle|\, \mathcal{F}_t\right) = (l_x - N_t) \int_t^T \mathbb{E}\left((\nu_u^L)^2 \,\middle|\, \mathcal{G}_t\right) {}_{u-t}p_{x+t}\mu_{x+u} \, du \quad (0 \le t \le T),$$

where $\nu_u^L := -(\tilde{S}_u^0)^{-1} F(u, \tilde{S}_u^1)_{T-u}p_{x+u}$. *For* $t = 0$ *it reduces to*

$$\mathbb{E}\left(\left(L_T^{H,\mathbb{Q}_T}\right)^2\right) = l_x \, {}_Tp_x \int_0^T \mathbb{E}\left(\left((\tilde{S}_u^0)^{-1} F(u, \tilde{S}_u^1)\right)^2\right) {}_{T-u}p_{x+u}\mu_{x+u} \, du.$$

Proof. Note, that $d\langle M \rangle_t = \rho_t \, dt$. Then we get with Itô's Isometry und Fubini's Theorem

$$\begin{aligned}
\mathbb{E}\left(\left(L_T^{H,\mathbb{Q}_T} - L_t^{H,\mathbb{Q}_T}\right)^2 \,\middle|\, \mathcal{F}_t\right) &= \mathbb{E}\left(\left(\int_t^T \nu_u^L \, dM_u\right)^2 \,\middle|\, \mathcal{F}_t\right) \\
&= \mathbb{E}\left(\int_t^T (\nu_u^L)^2 \rho_u \, du \,\middle|\, \mathcal{F}_t\right) \\
&= \int_t^T \mathbb{E}\left((\nu_u^L)^2 \,\middle|\, \mathcal{G}_t\right) \mathbb{E}(\rho_u \,|\, \mathcal{H}_t) \, du \\
&= \int_t^T \mathbb{E}\left((\nu_u^L)^2 \,\middle|\, \mathcal{G}_t\right) (l_x - N_t)_{u-t}p_{x+t}\mu_{x+u} \, du.
\end{aligned}$$

The special case $t = 0$ can be concluded with the observation ${}_up_x \, {}_{T-u}p_{x+u} = {}_Tp_x$. $\square$

Remark. In case one of the insured persons dies $(l_x - N_t)$ reduces by one. Thus, one unit of ${}_{T-t}p_{x+t}(\tilde{S}_t^0)^{-1} F(t, \tilde{S}_t^1)$ in the position $(\phi_t^0)^H$ is freed, which results in an immediate gain for the insurer. Also, the position in the risky asset is rather intuitive, since it is simply the Δ-hedge of the claim $f(\tilde{S}_T^1)$ times the expected value of survivors.

Corollary 3.2. *Suppose* K_T *is deterministic and* $H = (l_x - N_T)(\tilde{S}_T^0)^{-1} f(\tilde{S}_T^1)$. *Then the mean-variance optimal strategy* ξ^{V_0} *for a given inital capital* V_0 *is determined by*

$$\xi_t^{V_0} = \xi_t^{H,\mathbb{Q}_T} + \lambda_t \left(V_{t-}^{H,\mathbb{Q}_T} - V_0 - \int_0^t \xi_s^{V_0} \, dX_s\right) \quad (0 \le t \le T),$$

with minimal quadratic risk

$$J_0 = e^{-K_T} \left(\left(V_0^{H,\mathbb{Q}_T} - V_0\right)^2 + \mathbb{E}\left(\int_0^T e^{K_s} \, d[L^{H,\mathbb{Q}_T}]_s\right)\right).$$

In case we can choose the initial capital V_0, *then it is optimal to take* $V_0^{H,\mathbb{Q}_T}$.

Proof. Follows directly from the Equations (2.3.7) and (2.3.8). □

Let us now consider the standard Black-Scholes Model, where m, σ and r are constant. We will look at two examples of explict given functions f. One deals with a *pure unit-linked* contract and the other with a *unit-linked contract with guarantee.*

Example (**pure unit-linked**). Assume we have $f(s) = s$. This means, that the insured person obtains at maturity T the pure stock value $\tilde{S}^1_T$. The process $(F(t, \tilde{S}^1_t))_{0\le t\le T}$ is then determined by

$$F(t, \tilde{S}^1_t) = \mathbb{E}_{\mathbb{Q}_T}\left(e^{-r(T-t)}\tilde{S}^1_T \,\middle|\, \mathcal{G}_t\right) = \mathbb{E}_{\mathbb{Q}_T}\left(e^{rt}X_T \,\middle|\, \mathcal{G}_t\right) = e^{rt}X_t = \tilde{S}^1_t.$$

Thus, we have

$$\begin{aligned} V_t^{H,\mathbb{Q}_T} &= (l_x - N_t)_{T-t}p_{x+t}X_t, \\ V_0^{H,\mathbb{Q}_T} &= l_x\, {}_Tp_x\tilde{S}^1_0. \end{aligned}$$

Using $F_s(t, \tilde{S}^1_t) = 1$, the pseudo-optimal strategy is given by

$$\begin{aligned} \xi_t^H &= \xi_t^{H,\mathbb{Q}_T} = (l_x - N_{t-})_{T-t}p_{x+t} \\ (\phi_t^0)^H &= V_t^{H,\mathbb{Q}_T} - \xi_t^{H,\mathbb{Q}_T}X_t \\ &= (l_x - N_t)_{T-t}p_{x+t}X_t - (l_x - N_{t-})_{T-t}p_{x+t}X_t \\ &= -\Delta N_t\, {}_{T-t}p_{x+t}X_t. \end{aligned}$$

Finally, with $\nu_u^L = -X_u\, {}_{T-u}p_{x+u}$ the process $(L_t^{H,\mathbb{Q}_T})_{0\le t\le T}$ is given by

$$L_t^{H,\mathbb{Q}_T} = -\int_0^t X_u\, {}_{T-u}p_{x+u}\, dM_u,$$

and for the remaining risk we conclude

$$\mathbb{E}\left(\left(L_T^{H,\mathbb{Q}_T} - L_t^{H,\mathbb{Q}_T}\right)^2 \,\middle|\, \mathcal{F}_t\right) = (l_x - N_t)\int_t^T \mathbb{E}\left(X_u^2\, {}_{T-u}p_{x+u}^2 \,\middle|\, \mathcal{G}_t\right) {}_{u-t}p_{x+t}\mu_{x+u}\, du.$$

Example (**uni-linked with guarantee**). Assume we have $f(s) = \max(s, K)$, for some constant guarantee $K \ge 0$. Note, that in case $K = 0$ we have the same setting as in the previous example. Rewriting $f(s) = K + (s - K)_+$, we obtain the process $(F(t, \tilde{S}^1_t))_{0\le t\le T}$ by the use of the well known Black-Scholes formula

$$\begin{aligned} F(t, \tilde{S}^1_t) &= \mathbb{E}_{\mathbb{Q}_T}\left(e^{-r(T-t)}\left(K + \left(\tilde{S}^1_T - K\right)_+\right) \,\middle|\, \mathcal{G}_t\right) \\ &= Ke^{-r(T-t)} + \left(\tilde{S}^1_t\Phi(d_t) - Ke^{-r(T-t)}\Phi(d_t - \sigma\sqrt{T-t})\right) \\ &= Ke^{-r(T-t)}\Phi(-d_t + \sigma\sqrt{T-t}) + \tilde{S}^1_t\Phi(d_t), \end{aligned}$$

where Φ denotes the standard normal distribution and

$$d_t := \frac{\log \frac{\tilde{S}^1_t}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}.$$

Note that d_t depends on $\tilde{S}^1_t$. With $F_s(t, \tilde{S}^1_t) = \Phi(d_t)$, the pseudo-optimal strategy is given by

$$\begin{aligned}
\xi^{H,\mathbb{Q}_T}_t &= (l_x - N_{t-})_{T-t}p_{x+t}\Phi(d_t) \\
(\phi^0_t)^H &= (l_x - N_t)_{T-t}p_{x+t}e^{-rt}F(t, \tilde{S}^1_t) - (l_x - N_{t-})_{T-t}p_{x+t}\Phi(d_t)X_t \\
&= (l_x - N_t)_{T-t}p_{x+t}Ke^{-rT}\Phi(-d_t + \sigma\sqrt{T-t}) - \Delta N_{t\ T-t}p_{x+t}\Phi(d_t)X_t
\end{aligned}$$

and for its remaining risk we have with

$$\nu^L_u = -\left(Ke^{-rT}\Phi(-d_u + \sigma\sqrt{T-u}) + X_u\Phi(d_u)\right)_{T-u}p_{x+u},$$

that it is given by

$$\mathbb{E}\left(\left(L^{H,\mathbb{Q}_T}_T - L^{H,\mathbb{Q}_T}_t\right)^2 \middle| \mathcal{F}_t\right) = (l_x - N_t)\int_t^T \mathbb{E}\left(\left(\nu^L_u\right)^2 \middle| \mathcal{G}_t\right)_{u-t}p_{x+t}\mu_{x+u}\,du.$$

4 Conclusion

We started under the simplified case where the underlying discounted, d-dimensional, real valued price process X is already a local martingale under $\mathbb{P}$. By relying on the terminal constraint $V_T = H$ $\mathbb{P}$-a.s., where H is a given discounted contingent claim, our variance-minimization problem was given by

$$\min_{\phi} \mathbb{E}\left((C_T(\phi) - \mathbb{E}(C_T(\phi)))^2\right),$$

where $\phi = (\phi^0, \xi)$ runs over all H-admissible strategies. Under the assumption $H \in L^2(\mathcal{F}_T, \mathbb{P})$, we used the **Galtchouk-Kunita-Watanabe decomposition**

$$H = \mathbb{E}(H) + \int_0^T \xi^*_s \cdot dX_s + L^*_T \quad \mathbb{P}\text{-a.s.},$$

where $\xi^* \in L^2(X)$ and $L^* \in \mathcal{M}^2(\mathbb{P})$, with $\mathbb{E}(L^*_0) = 0$, is strongly orthogonal to $\mathcal{I}^2(X)$. By Theorem 2.3 the optimal strategy was directly given by ξ^*. But the position ϕ^0 in the bank account only needed to fulfill the admissibility condition at maturity.
To get a better criterion for ϕ^0, we switched to the idea of minimizing the conditional mean squared error

$$R_t(\phi) := \mathbb{E}\left((C_T(\phi) - C_t(\phi))^2 \,\middle|\, \mathcal{F}_t\right) \quad (0 \le t \le T).$$

With the help of Lemma 2.4 we could restrict our search for a risk-minimizing strategy to mean self-financing strategies. Thus, $R_t(\phi)$ is in fact the conditional variance of the terminal costs. Theorem 2.5 then gave us the unique risk minimizing strategy where $\xi = \xi^*$ and ϕ^0 is chosen in such a way that the strategy maintains its mean self-financing property for each $0 \le t \le T$.

In the general semimartingale case, we tried first to find a risk-minimizing strategy. Unfortunately, due to a time inconsistency problem this is not possible as it is shown in Theorem 2.7. Hence, we switched to Definition 2.8 of a *local* risk-minimizing strategy. Subsequently, we tried to characterize the local risk-minimizing strategy by a variational argument. The idea was that any small pertubation of the optimal strategy should lead to an (asymptotically) increased risk. The actual calculation became quite technical, where X needed to fulfill a certain **structure condition**, see Definition 2.9, to get some meaningful results. Finally, Theorem 2.11 gave us a characterization of a locally risk-minimizing strategy. Theorem 2.12 then linked the locally risk-minimizing strategy to the integrand of the **Föllmer-Schweizer decomposition**

$$H = \mathbb{E}(H) + \int_0^T \xi_s^H \cdot dX_s + L_T^H \quad \mathbb{P}\text{-a.s.},$$

where $\xi^H \in \Xi$ and $L^H \in \mathcal{M}^2(\mathbb{P})$, with $\mathbb{E}(L_0^H) = 0$, is strongly orthogonal to $\mathcal{I}^2(M)$ with respect to $\mathbb{P}$. Since the Galtchouk-Kunita-Watanabe decomposition yielded a risk-minimizing strategy in the martingale case, we already suspected that the Föllmer-Schweizer decomposition yields a locally risk-minimizing strategy. However, it took quite some technical effort to verify this idea.

Subsequently, we proved the existence and uniqueness of the Föllmer-Schweizer decomposition under the assumption, that X satisfies the **structure condition** and that the **mean-variance tradeoff process** K is uniformly bounded, see Definition 2.9. This seems quite restrictive, but in fact these assumptions are quite natural in arbitrage free time continuous market models: One often prefers to be able to apply Girsanov's Theorem and switch to an equivalent martingale measure $\mathbb{Q}$ under which X is a local martingale. Furthermore, in these models K is usually the integrated squared *market price of risk*.

The existence and uniqueness Theorem 2.16 of the Föllmer-Schweizer decomposition took again quite some effort, since in the literature a different formulation of the Föllmer-Schweizer decomposition was used, namely

$$H = H_0 + \int_0^T \xi_s^H \cdot dX_s + \bar{L}_T^H \quad \mathbb{P}\text{-a.s.},$$

where $H_0 := \mathbb{E}(H \,|\, \mathcal{F}_0) \in L^2(\mathcal{F}_0, \mathbb{P})$, $\xi^H \in \Xi$ and $\bar{L}^H \in \mathcal{M}_0^2(\mathbb{P})$ is strongly orthogonal to $\mathcal{I}^2(M)$. The connection to our introduced decomposition can be seen by

$$H = \underbrace{\mathbb{E}(H) + L_0^H}_{= H_0} + \int_0^T \xi_s^H \cdot dX_s + \underbrace{L_T^H - L_0^H}_{= \bar{L}_T^H} \quad \mathbb{P}\text{-a.s.},$$

where one shifts the initial random value L_0^H to the $\mathcal{F}_0$-measurable random variable H_0. Thus, we had to be very careful in our argumentation regarding the starting value.

As a next step we tried to find the Föllmer-Schweizer decomposition of H. The idea was to switch with Girsanov's Theorem to an equivalent martingale measure $\mathbb{Q}$, then find its Galtchouk-Kunita-Watanabe decomposition under $\mathbb{Q}$ and finally, switch back to the measure $\mathbb{P}$. Since we are in an incomplete market, the question was which equivalent martingale measure to choose. Of course, under each $\mathbb{Q} \in \mathcal{Q}$, where $\mathcal{Q}$ denotes the set of equivalent local martingale measures, the Galtchouk-Kunita-Watanabe decomposition yields the optimal strategy, which minimizes

$$\mathbb{E}_{\mathbb{Q}}\left((C_T(\phi) - C_t(\phi))^2 \,\middle|\, \mathcal{F}_t \right) \quad (0 \le t \le T).$$

But here we want to find the risk-minmizing strategy under $\mathbb{P} \notin \mathcal{Q}$. An essential ingrediant in the proof of Theorem 2.5 was the orthogonality of L^* and X. Thus, by switching back and forth between $\mathbb{P}$ and $\mathbb{Q}$ we may loose this property. Fortunately, the **minimal martingale measure** $\mathbb{Q}_T$ maintains this property and it is unique under the assumption of X beeing continuous, see Theorem 2.18. Furthermore, the technical Theorem 2.17 proved that the continuity of X implies the structure condition. Hence, for **continuous** X and **uniformly bounded** K, a locally risk-minimizing strategy can be obtained by finding the Galtchouk-Kunita-Watanabe decomposition of H under the minimal martingale measure $\mathbb{Q}_T$. This is useful, since the density process Z_T of $\mathbb{Q}_T$ is explicitly given by

$$\frac{d\mathbb{Q}_T}{d\mathbb{P}} := Z_T = \mathcal{E}\left(-\int \lambda \cdot dM\right)_T = \exp\left(-\int_0^T \lambda_s \cdot dM_s - \frac{1}{2}K_T\right)$$

and we can study the behaviour of X under $\mathbb{Q}_T$.

As a second idea we relied on the self-financing constraint and tried to minimize the mean-squared error of the terminal portfolio value

$$\min_{\xi \in \Xi} \mathbb{E}\left(\left(H - V_0 - \int_0^T \xi_s \cdot dX_s\right)^2\right),$$

where $V_0 \in \mathbb{R}$ is the given initial capital. Since ϕ is self-financing, the strategy is uniquely determined by the choice of $(V_0, \xi) \in \mathbb{R} \times \Xi$ (compare Equation (1.1.2)). To find its solution we had to project $H \in L^2(\mathcal{F}_T, \mathbb{P})$ onto the linear space

$$G_T(\Xi) = \left\{ \int_0^T \xi_s \cdot dX_s \,\middle|\, \xi \in \Xi \right\}.$$

In the martingale case this was easy, since by construction we know that the stochastic integral with respect to a local martingale is an isometry and hence $G_T(\Xi)$ is closed in $L^2(\mathbb{P})$. Consequently, by Hilbert's Projection Theorem the solution exists and is unique. Furthermore, in the martingale case the solution was again given by the integrand of the Galtchouk-Kunita-Watanabe decomposition.

In the general semimartingale case the proof of $G_T(\Xi)$ beeing a closed subspace took quite some effort. We showed this in Theorem 2.25 with the help of the Föllmer-Schweizer decomposition, which is continuous under the assumptions of X satisfying the structure condition and K beeing uniformly bounded, see Theorem 2.24. Using Hilbert's Projection Theorem again, we obtained the existence and uniqueness of the mean-variance optimal strategy.

In practice, we would like to have a more explicit representation of the mean-variance optimal trading strategy. Thus we looked at the special case (2.3.4) and assumed that X is continuous and K is uniformly bounded. Then, Theorem 2.28 gave us a mean-variance optimal strategy in feedback form.

Lastly, we pointed out that if one wants to find a general explicit representation of the mean-variance optimal trading strategy, one has to work with the **variance optimal martingale measure** $\mathbb{Q}_T^{opt}$. This we did not study. But in the literature [Delbaen and Schachermayer, 1996] it is proven that our special case (2.3.4) is always fulfilled under the variance optimal martingale measure and that in this case the variance optimal martingale measure and the minimal martingale measure coincide.

In Chapter three, which is based on [Møller, 1998], we applied the introduced theory of quadratic hedging to a portfolio of unit-linked life insurance contracts. The key difference to [Møller, 1998] was that we solved the problem of local risk-minimization under $\mathbb{P} \notin \mathcal{Q}$ instead of $\mathbb{Q} \in \mathcal{Q}$.

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