Rules for asymptotic computations

In[ ]:=

```
PoincareSumCollect :=
  {PoincareSum[a_, {m, 1, ∞}] + PoincareSum[b_, {m, 1, ∞}] → PoincareSum[a + b, {m, 1, ∞}],
   PoincareSum[a_, {m, 1, ∞}] - PoincareSum[b_, {m, 1, ∞}] →
    PoincareSum[a - b, {m, 1, ∞}]};

PoincareSumFactorUnderSum := a_ PoincareSum[b_, {m, 1, ∞}] → PoincareSum[a b, {m, 1, ∞}];

PoincareSumIndexShiftUp[Δm_] :=
  PoincareSum[a_, {m, 1, ∞}] :→ PoincareSum[a /. m → m + Δm, {m, 1 - Δm, ∞}];

PoincareSumSplitOffTerms[Δm_] := PoincareSum[a_, {m_, m1_, ∞}] :→
   Sum[a, {m, m1, m1 + Δm - 1}] + PoincareSum[a, {m, m1 + Δm, ∞}];

PoincareSumNormalize[K_] := PoincareSum[a_, {m, 1, ∞}] :→ Sum[a, {m, 1, K}];
```

Appendix A

In[ ]:=

```
AsymptoticsLogGamma[a_, x_] :=
```

$$\left(x + a - \frac{1}{2}\right) \mathrm{Log}[x] - x + \frac{1}{2} \mathrm{Log}[2\,\pi] - \mathrm{PoincareSum}\left[\frac{(-1)^{m-1}}{m} \mathrm{HurwitzZeta}[-m,\, a]\, x^{-m},\, \{m,\, 1,\, \infty\}\right];$$

```
AsymptoticsHurwitzZetaPrime[a_, x_] :=
```

$$\frac{1}{2} x^2 \mathrm{Log}[x] - \frac{1}{4} x^2 - \mathrm{HurwitzZeta}[0,\, a]\, x\, \mathrm{Log}[x] - \mathrm{HurwitzZeta}[-1,\, a]\, \mathrm{Log}[x] -$$

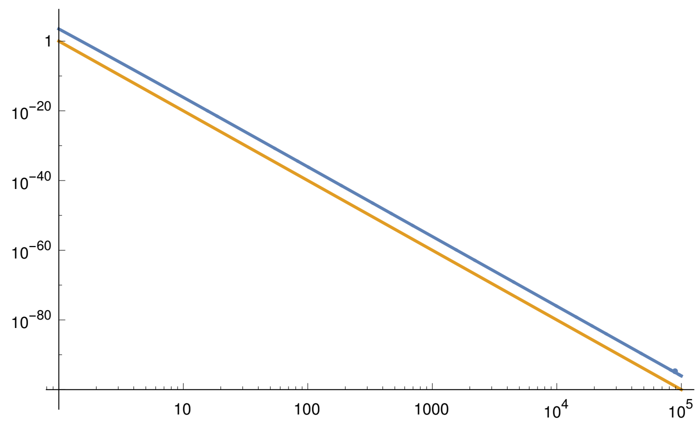
$$\mathrm{HurwitzZeta}[-1,\, a] + \mathrm{PoincareSum}\left[\frac{(-1)^{m}}{m\,(m+1)} \mathrm{HurwitzZeta}[-m-1,\, a]\, x^{-m},\, \{m,\, 1,\, \infty\}\right];$$

Cross Checks

Basic LogGamma Asymptotics

In[ ]:=
```
f = Log[Gamma[x + a]];

K = 20; (* order of error and number of terms minus 1 in sum *)
a = 2 Sqrt[2]; (* cf. paper *)

REF = (x + a - 1/2) Log[x] - x + 1/2 Log[2 π] - Sum[(-1)^(m-1)/m HurwitzZeta[-m, a] x^-m, {m, 1, K - 1}];

LogLogPlot[{Abs[f - REF], x^-K}, {x, 1, 100 000}, WorkingPrecision → 128]

Clear[a, f, K];
```

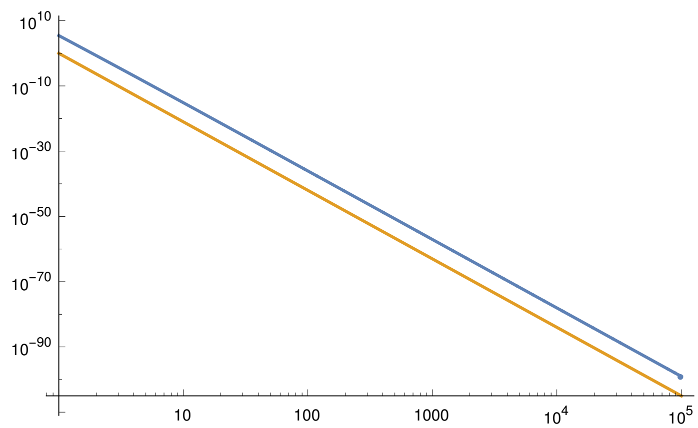
Out[ ]=



The s-derivative of the Hurwitz zeta function

In[ ]:=
```
f = D[HurwitzZeta[s, x + a], s];

K = 21;(* order of error and number of terms minus 1 in sum *)
s = -1;(* DO NOT CHANGE; cf. paper *)
a = 2 Sqrt[3] ;(* cf. paper *)

REF = 1/2 x^2 Log[x] - 1/4 x^2 - HurwitzZeta[0, a] x Log[x] - HurwitzZeta[-1, a] Log[x] -
    HurwitzZeta[-1, a] + Sum[(-1)^m/(m (m + 1)) HurwitzZeta[-m - 1, a] x^-m, {m, 1, K - 1}];

LogLogPlot[{Abs[f - REF], x^-K}, {x, 1, 100 000}, WorkingPrecision → 128]

Clear[a, f, K, s];
```

Out[ ]=



Section 2

Lemma 2.1

Log $\lambda_n^{(\alpha,\beta)}$ asymptotics

Cross Check of formula

```
λn[n_, α_, β_] := 2^-n Binomial[2 n + α + β, n];

n = 13; (* test integer *)
Limit[JacobiP[n, α, β, x] / (λn[n, α, β] x^n), x → ∞] // FullSimplify
Clear[n];
```

Out[ ]= 1

Asymptotic Relation

Verification of starting point

```
In[ ]:= REF = Log[2^-n Binomial[2 n + α + β, n]];

RES = - Log[2] n + Log[Gamma[2 n + α + β + 1]] - Log[Gamma[n + α + β + 1]] - Log[Gamma[n + 1]];

(* direct symbolic verification *)
REF == RES // FullSimplify[#, Assumptions → {n ∈ Integers, n ≥ 1, α > -1, β > -1}] &

(* special case *)
n = 0;
REF == RES // FullSimplify

Clear[n];
```

Out[ ]= True

Out[ ]= True

Application of LogGamma asymptotics

In[ ]:= **TMP01 = - Log[2] n + AsymptoticsLogGamma[α + β + 1, 2 n] - AsymptoticsLogGamma[α + β + 1, n] - AsymptoticsLogGamma[1, n]**

Out[ ]= $-n\,\text{Log}[2]-\left(\frac{1}{2}+n\right)\text{Log}[n]-\left(\frac{1}{2}+n+\alpha+\beta\right)\text{Log}[n]+\left(\frac{1}{2}+2\,n+\alpha+\beta\right)\text{Log}[2\,n]-\frac{1}{2}\text{Log}[2\,\pi]+\text{PoincareSum}\left[\frac{(-1)^{-1+m}\,n^{-m}\,\text{HurwitzZeta}[-m,1+\alpha+\beta]}{m},\{m,1,\infty\}\right]-\text{PoincareSum}\left[\frac{(-1)^{-1+m}\,2^{-m}\,n^{-m}\,\text{HurwitzZeta}[-m,1+\alpha+\beta]}{m},\{m,1,\infty\}\right]+\text{PoincareSum}\left[\frac{(-1)^{-1+m}\,n^{-m}\,\text{Zeta}[-m]}{m},\{m,1,\infty\}\right]$

Simplifications

In[ ]:= **TMP01 //. PoincareSumCollect**

Out[ ]= $-n\,\text{Log}[2]-\left(\frac{1}{2}+n\right)\text{Log}[n]-\left(\frac{1}{2}+n+\alpha+\beta\right)\text{Log}[n]+\left(\frac{1}{2}+2\,n+\alpha+\beta\right)\text{Log}[2\,n]-\frac{1}{2}\text{Log}[2\,\pi]+\text{PoincareSum}\left[\frac{(-1)^{-1+m}\,n^{-m}\,\text{HurwitzZeta}[-m,1+\alpha+\beta]}{m}+\frac{\left(-\frac{1}{2}\right)^{m}\,n^{-m}\,\text{HurwitzZeta}[-m,1+\alpha+\beta]}{m}+\frac{(-1)^{-1+m}\,n^{-m}\,\text{Zeta}[-m]}{m},\{m,1,\infty\}\right]$

In[ ]:= $\textbf{TMP} = \frac{(-1)^{-1+m}\,n^{-m}\,\text{HurwitzZeta}[-m,1+\alpha+\beta]}{m}+\frac{\left(-\frac{1}{2}\right)^{m}\,n^{-m}\,\text{HurwitzZeta}[-m,1+\alpha+\beta]}{m}+\frac{(-1)^{-1+m}\,n^{-m}\,\text{Zeta}[-m]}{m}$ **// PowerExpand // Factor**

Out[ ]= $\frac{1}{m}(-1)^{1+m}\,2^{-m}\,n^{-m}\left(-\text{HurwitzZeta}[-m,1+\alpha+\beta]+2^{m}\,\text{HurwitzZeta}[-m,1+\alpha+\beta]+2^{m}\,\text{Zeta}[-m]\right)$

In[ ]:= $\textbf{TMP} == \frac{(-1)^{m-1}}{m}\left(\left(1-2^{-m}\right)\text{HurwitzZeta}[-m,1+\alpha+\beta]+\text{Zeta}[-m]\right)n^{-m}$ **// FullSimplify**

Out[ ]= True

In[ ]:=
$$\text{TMP02} = -n\,\text{Log}[2] - \left(\frac{1}{2}+n\right)\text{Log}[n] - \left(\frac{1}{2}+n+\alpha+\beta\right)\text{Log}[n] + \left(\frac{1}{2}+2\,n+\alpha+\beta\right)\text{Log}[2\,n] - \frac{1}{2}\text{Log}[2\,\pi] + \text{PoincareSum}\left[\frac{(-1)^{m-1}}{m}\left(\left(1-2^{-m}\right)\text{HurwitzZeta}[-m,\,1+\alpha+\beta]+\text{Zeta}[-m]\right)n^{-m},\,\{m,\,1,\,\infty\}\right];$$

In[ ]:=
$$-n\,\text{Log}[2] - \left(\frac{1}{2}+n\right)\text{Log}[n] - \left(\frac{1}{2}+n+\alpha+\beta\right)\text{Log}[n] + \left(\frac{1}{2}+2\,n+\alpha+\beta\right)\text{Log}[2\,n] - \frac{1}{2}\text{Log}[2\,\pi]\;//\;\text{FullSimplify}$$

Out[ ]=
$$(n+\alpha+\beta)\,\text{Log}[2] - \frac{1}{2}\text{Log}[n\,\pi]$$

In[ ]:=
$$\text{REF} = (n+\alpha+\beta)\,\text{Log}[2] - \frac{1}{2}\text{Log}[n\,\pi];$$

$$\text{RES} = \text{Log}[2]\,n - \frac{1}{2}\text{Log}[n] + (\alpha+\beta)\,\text{Log}[2] - \frac{1}{2}\text{Log}[\pi];$$

```
REF == RES // FullSimplify
```

Out[ ]= True

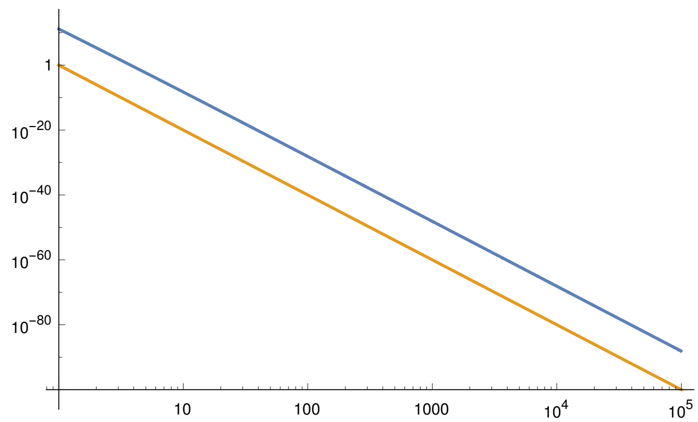
Formula

In[ ]:=
$$\text{ASYMP}\lambda[n\_,\,\alpha\_,\,\beta\_] := \text{Log}[2]\,n - \frac{1}{2}\text{Log}[n] + (\alpha+\beta)\,\text{Log}[2] - \frac{1}{2}\text{Log}[\pi] + \text{PoincareSum}\left[\frac{(-1)^{m-1}}{m}\left(\left(1-2^{-m}\right)\text{HurwitzZeta}[-m,\,1+\alpha+\beta]+\text{Zeta}[-m]\right)n^{-m},\,\{m,\,1,\,\infty\}\right];$$

Cross Check

In[∘]:=
```
REF = − Log[2] n + Log[Gamma[2 n + α + β + 1]] − Log[Gamma[n + α + β + 1]] − Log[Gamma[n + 1]];

K = 20;
α = √2 ;
β = π;

ASYMP = ASYMPλ[n, α, β] /. PoincareSumNormalize[K − 1];

LogLogPlot[{Abs[REF − ASYMP], n^-K}, {n, 1, 100 000}, WorkingPrecision → 128]

Clear[α, β, K];
```

Out[∘]=



Log $P_n^{(\alpha,\beta)}(1)$ asymptotics

Cross Check of formula

In[∘]:=
```
JacobiP[n, α, β, 1] == Pochhammer[α + 1, n] / n! // FullSimplify

Pochhammer[α + 1, n] / n! == 1 / Gamma[α + 1] * Gamma[n + α + 1] / Gamma[n + 1] // FullSimplify
```

Out[∘]= True

Out[∘]= True

Asymptotic Relation

Verification of starting point

In[ ]:=
```
REF = Log[Pochhammer[α + 1, n] / n!];

RES = - Log[Gamma[α + 1]] + Log[Gamma[n + α + 1]] - Log[Gamma[n + 1]];

(* direct symbolic verification *)
REF == RES // FullSimplify[#, Assumptions → {n ∈ Integers, n ≥ 0, α > -1, β > -1}] &
```

Out[ ]= True

Application of LogGamma asymptotics

In[ ]:=
```
TMP01 = - Log[Gamma[α + 1]] + AsymptoticsLogGamma[α + 1, n] - AsymptoticsLogGamma[1, n]
```

Out[ ]=
$$-\left(\frac{1}{2}+n\right)\mathrm{Log}[n]+\left(\frac{1}{2}+n+\alpha\right)\mathrm{Log}[n]-\mathrm{Log}[\mathrm{Gamma}[1+\alpha]]-\mathrm{PoincareSum}\left[\frac{(-1)^{-1+m}\,n^{-m}\,\mathrm{HurwitzZeta}[-m,\,1+\alpha]}{m},\,\{m,\,1,\,\infty\}\right]+\mathrm{PoincareSum}\left[\frac{(-1)^{-1+m}\,n^{-m}\,\mathrm{Zeta}[-m]}{m},\,\{m,\,1,\,\infty\}\right]$$

Simplifications

In[ ]:=
```
TMP01 //. PoincareSumCollect
```

Out[ ]=
$$-\left(\frac{1}{2}+n\right)\mathrm{Log}[n]+\left(\frac{1}{2}+n+\alpha\right)\mathrm{Log}[n]-\mathrm{Log}[\mathrm{Gamma}[1+\alpha]]+\mathrm{PoincareSum}\left[\frac{(-1)^{m}\,n^{-m}\,\mathrm{HurwitzZeta}[-m,\,1+\alpha]}{m}+\frac{(-1)^{-1+m}\,n^{-m}\,\mathrm{Zeta}[-m]}{m},\,\{m,\,1,\,\infty\}\right]$$

In[ ]:=
$$\mathrm{TMP} = \frac{(-1)^{m}\,n^{-m}\,\mathrm{HurwitzZeta}[-m,\,1+\alpha]}{m}+\frac{(-1)^{-1+m}\,n^{-m}\,\mathrm{Zeta}[-m]}{m}\ \text{// Factor}$$

Out[ ]=
$$\frac{(-1)^{m}\,n^{-m}\,(\mathrm{HurwitzZeta}[-m,\,1+\alpha]-\mathrm{Zeta}[-m])}{m}$$

In[ ]:= TMP == $\frac{(-1)^m}{m}$ (HurwitzZeta[-m, 1 + α] - Zeta[-m]) n^{-m} // FullSimplify

Out[ ]= True

In[ ]:= TMP02 = $-\left(\frac{1}{2}+n\right)$ Log[n] + $\left(\frac{1}{2}+n+\alpha\right)$ Log[n] - Log[Gamma[1 + α]] +
PoincareSum[$\frac{(-1)^m}{m}$ (HurwitzZeta[-m, 1 + α] - Zeta[-m]) n^{-m}, {m, 1, ∞}];

In[ ]:= $-\left(\frac{1}{2}+n\right)$ Log[n] + $\left(\frac{1}{2}+n+\alpha\right)$ Log[n] - Log[Gamma[1 + α]] // FullSimplify

Out[ ]= α Log[n] - Log[Gamma[1 + α]]

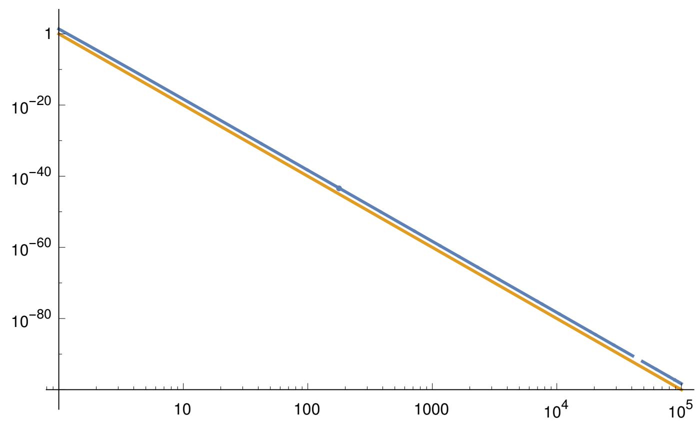
Formula

In[ ]:= ASYMPJacobiPofOne[n_, α_, β_] := α Log[n] - Log[Gamma[1 + α]] +
PoincareSum[$\frac{(-1)^m}{m}$ (HurwitzZeta[-m, 1 + α] - Zeta[-m]) n^{-m}, {m, 1, ∞}];

Cross Check

In[ ]:=
```
REF = - Log[Gamma[α + 1]] + Log[Gamma[n + α + 1]] - Log[Gamma[n + 1]];

K = 20;
α = √2 ;
β = π;

ASYMP = ASYMPJacobiPofOne[n, α, β] /. PoincareSumNormalize[K - 1];

LogLogPlot[{Abs[REF - ASYMP], n^-K}, {n, 1, 100 000}, WorkingPrecision → 256]

Clear[α, β, K];
```

Out[ ]=



Lemma 2.4

Log $D_n^{(\alpha,\beta)}$ asymptotics

Cross Check of formula

Symbolic Cross Check (small degree n)

In[ ]:=
```
n = 4;
α = 1 / 2;
β = π;

(* Thm 2.3 *)
REF = 2^(-n (n-1)) Product[ν^(ν-2 n+2) (ν + α)^(ν-1) (ν + β)^(ν-1) (ν + n + α + β)^(n-ν), {ν, 1, n}] /
    (2^-n Binomial[2 n + α + β, n])^(2 n-2) // FullSimplify;

(* definition of discriminant *)
zeros = x /. Solve[JacobiP[n, α, β, x] == 0, x];

RES = Product[(zeros[[j]] - zeros[[k]])^2, {j, 1, n - 1}, {k, j + 1, n}] // Simplify;

(* verification *)
REF == RES // FullSimplify

Clear[α, β, n];
```

Out[ ]= True

Numerical Cross Check (general degree n)

In[ ]:=
```
n = 7;
α = 1 / 2;
β = π;

(* Thm 2.3 *)
REF = 2^(-n (n-1)) Product[ν^(ν-2 n+2) (ν + α)^(ν-1) (ν + β)^(ν-1) (ν + n + α + β)^(n-ν), {ν, 1, n}] /
    (2^-n Binomial[2 n + α + β, n])^(2 n-2) // FullSimplify;

(* definition of discriminant *)
zeros = x /. NSolve[JacobiP[n, α, β, x] == 0, x, WorkingPrecision → 64];

RES = Product[(zeros[[j]] - zeros[[k]])^2, {j, 1, n - 1}, {k, j + 1, n}] // Simplify;

(* verification *)
RES - REF

Clear[α, β, n];
```

Out[ ]= $0. \times 10^{-70}$

Direct computation of discriminant using Thm 2.3

In[ ]:=
```
REFDiscriminant = 2^(-n (n-1)) Product[ν^(ν-2 n+2) (ν + α)^(ν-1) (ν + β)^(ν-1) (ν + n + α + β)^(n-ν), {ν, 1, n}]
```

Out[ ]= $2^{-(-1+n)\,n}\, e^{-\frac{1}{12}-2\,n\,\mathrm{Log[Gamma[1+n]]}+\mathrm{Zeta}^{(1,0)}[-1,1+n]-\mathrm{Zeta}^{(1,0)}[-1,2+\alpha]+\mathrm{Zeta}^{(1,0)}[-1,1+n+\alpha]-\mathrm{Zeta}^{(1,0)}[-1,2+\beta]+\mathrm{Zeta}^{(1,0)}[-1,1+n+\beta]+\mathrm{Zeta}^{(1,0)}[-}$

$\mathrm{Glaisher\ Gamma}[1+n]^2$

In[ ]:=
```
TMP =
  Log[REFDiscriminant] //. {Log[A_ B_] :→ Log[A] + Log[B], Log[A_^B_] :→ B Log[A]} // FullSimplify
```

Out[ ]= $-\frac{1}{12} - (-1+n)\,n\,\mathrm{Log}[2] + \mathrm{Log[Glaisher]} + 2\,\mathrm{Log[Gamma[1+n]]} -$
$2\,n\,\mathrm{Log[Gamma[1+n]]} + \mathrm{Zeta}^{(1,0)}[-1,\ 1+n] - \mathrm{Zeta}^{(1,0)}[-1,\ 2+\alpha] +$
$\mathrm{Zeta}^{(1,0)}[-1,\ 1+n+\alpha] - \mathrm{Zeta}^{(1,0)}[-1,\ 2+\beta] + \mathrm{Zeta}^{(1,0)}[-1,\ 1+n+\beta] +$
$\mathrm{Zeta}^{(1,0)}[-1,\ 1+n+\alpha+\beta] - \mathrm{Zeta}^{(1,0)}[-1,\ 1+2\,n+\alpha+\beta] + (1+\alpha)\,\mathrm{Zeta}^{(1,0)}[0,\ 2+\alpha] -$
$(1+\alpha)\,\mathrm{Zeta}^{(1,0)}[0,\ 1+n+\alpha] + (1+\beta)\,\mathrm{Zeta}^{(1,0)}[0,\ 2+\beta] - (1+\beta)\,\mathrm{Zeta}^{(1,0)}[0,\ 1+n+\beta] -$
$(2\,n+\alpha+\beta)\,\mathrm{Zeta}^{(1,0)}[0,\ 1+n+\alpha+\beta] + (2\,n+\alpha+\beta)\,\mathrm{Zeta}^{(1,0)}[0,\ 1+2\,n+\alpha+\beta]$

In[ ]:=
```
-1/12 + Log[Glaisher] == -Zeta^(1,0)[-1, 1] == -Zeta'[-1] // FullSimplify
```
Out[ ]= True

In[ ]:=
```
TMP == -(-1 + n) n Log[2] + Zeta^(1,0)[-1, 1 + n] - Zeta'[-1] - 2 (n - 1) Log[Gamma[1 + n]] +
    Zeta^(1,0)[-1, 1 + n + α] - Zeta^(1,0)[-1, 2 + α] + (1 + α) Zeta^(1,0)[0, 2 + α] -
    (1 + α) Zeta^(1,0)[0, 1 + n + α] + Zeta^(1,0)[-1, 1 + n + β] - Zeta^(1,0)[-1, 2 + β] +
    (1 + β) Zeta^(1,0)[0, 2 + β] - (1 + β) Zeta^(1,0)[0, 1 + n + β] + Zeta^(1,0)[-1, 1 + n + α + β] -
    Zeta^(1,0)[-1, 1 + 2 n + α + β] - (2 n + α + β) Zeta^(1,0)[0, 1 + n + α + β] +
    (2 n + α + β) Zeta^(1,0)[0, 1 + 2 n + α + β] // FullSimplify
```
Out[ ]= True

Not used, since no proof for this result.

Starting Point of Proof of Lemma 2.4

In[ ]:=
```
Sum[(k + x + a)^-s, {k, m + 1, n}] == HurwitzZeta[s, m + x + a + 1] - HurwitzZeta[s, n + x + a + 1]
(* follows from definition of Hurwitz zeta function *)
```
Out[ ]= True

In[ ]:=
```
Sum[(k + x + a) Log[k + x + a], {k, m + 1, n}] == Zeta^(1,0)[-1, n + x + a + 1] - Zeta^(1,0)[-1, m + x + a + 1]
(* follows from termwise differentiation w.r.t. s of the sum and setting s to -1. *)
```
Out[ ]= True

$\mathfrak{A}_n$

In[ ]:=
```
FracA[n_] := Sum[(v - 2 n + 2) Log[v], {v, 1, n}];
FracA[n] // Distribute
```
Out[ ]=
```
-1/12 + Log[Glaisher] + 2 Log[Gamma[1 + n]] - 2 n Log[Gamma[1 + n]] + Zeta^(1,0)[-1, 1 + n]
```

In[ ]:=
```
REF = FracA[n];

RES = Zeta^(1,0)[-1, n + 1] - Zeta'[-1] - 2 (n - 1) Log[Gamma[1 + n]];

REF == RES // FullSimplify
```
Out[ ]= True

In[ ]:=
```
FracA2[n_] := Zeta^(1,0)[-1, n + 1] - Zeta'[-1] - 2 (n - 1) Log[Gamma[1 + n]];
```

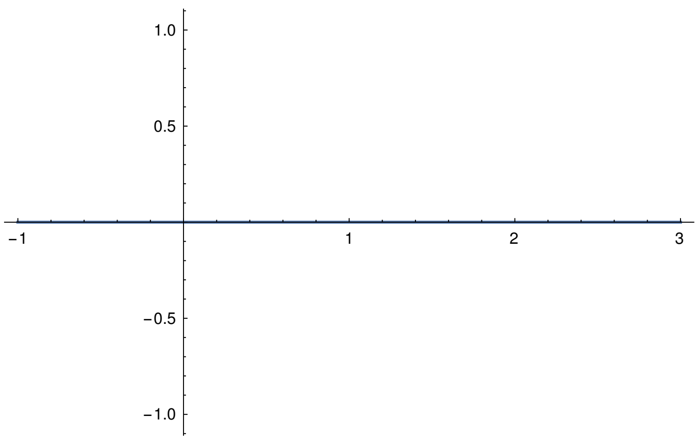
$\mathcal{B}_n(\alpha)$

In[ ]:=
```
FracB[n_, α_] := Sum[(v - 1) Log[v + α], {v, 1, n}];
FracB[n, α] // Simplify
```

Out[ ]=
```
-Zeta^(1,0)[-1, 2 + α] + Zeta^(1,0)[-1, 1 + n + α] + (1 + α) (Zeta^(1,0)[0, 2 + α] - Zeta^(1,0)[0, 1 + n + α])
```

```
n = 3; (* positive integer *)

REF = FracB[n, α];

RES = Zeta^(1,0)[-1, n + α + 1] - Zeta^(1,0)[-1, α + 1] - (α + 1) Log[Pochhammer[α + 1, n]];

f = REF - RES // FullSimplify[#, Assumptions → {α > -1}] &

Plot[f, {α, -1, 3}, WorkingPrecision → 64]

Clear[n, f];
```

Out[ ]=
```
Log[2 + α] + 2 Log[3 + α] + (1 + α) Log[(1 + α) (2 + α) (3 + α)] + Zeta^(1,0)[-1, 1 + α] - Zeta^(1,0)[-1, 4 + α]
```

Out[ ]=


Proof straight forward; Verification via Mathematica via selected examples.

In[ ]:=
```
FracB2[n_, α_] := Zeta^(1,0)[-1, n + α + 1] - Zeta^(1,0)[-1, α + 1] - (α + 1) Log[Pochhammer[α + 1, n]];
```

C_n (b)

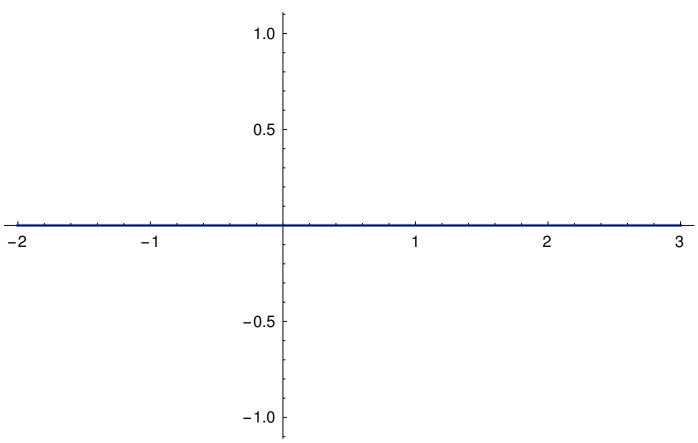
In[ ]:=
```
FracC[n_, b_] := Sum[(n - v) Log[v + n + b], {v, 1, n}];
FracC[n, b] // FullSimplify
```

Out[ ]=
```
Zeta^(1,0)[-1, 1 + b + n] - Zeta^(1,0)[-1, 1 + b + 2 n] -
  (b + 2 n) (Zeta^(1,0)[0, 1 + b + n] - Zeta^(1,0)[0, 1 + b + 2 n])
```

In[ ]:=
```
n = 2; (* positive integer *)

REF = FracC[n, b];

RES = (2 n + b) Log[Pochhammer[n + b + 1, n]] - Zeta^(1,0)[-1, 2 n + b + 1] + Zeta^(1,0)[-1, n + b + 1];

f = REF - RES // FullSimplify[#, Assumptions → {b > -2}] &

Plot[f, {b, -2, 3}, WorkingPrecision → 64]

Clear[n, f];
```

Out[ ]=
```
Log[3 + b] - (4 + b) Log[(3 + b) (4 + b)] - Zeta^(1,0)[-1, 3 + b] + Zeta^(1,0)[-1, 5 + b]
```

Out[ ]=



Proof straight forward; Verification via Mathematica via selected examples.

In[ ]:=
```
FracC2[n_, b_] :=
  (2 n + b) Log[Pochhammer[n + b + 1, n]] - Zeta^(1,0)[-1, 2 n + b + 1] + Zeta^(1,0)[-1, n + b + 1];
```

Asymptotic Relation

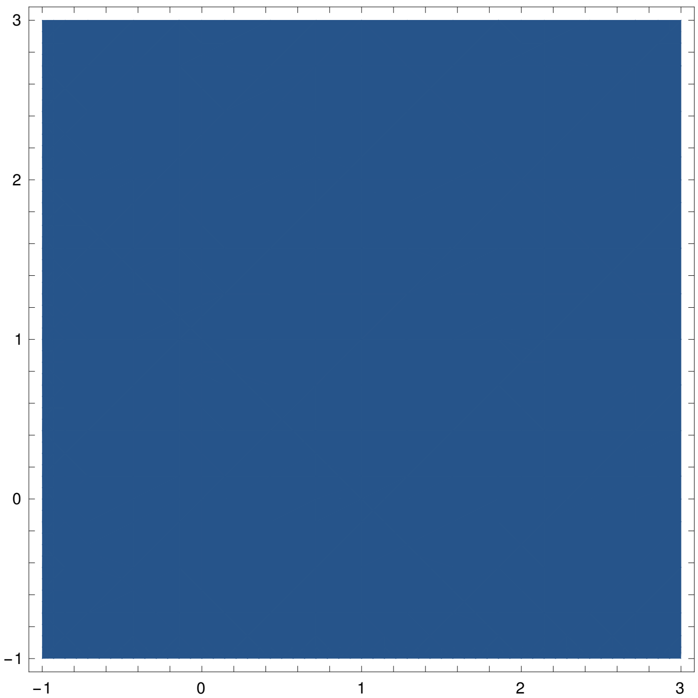
Verification of starting point

In[ ]:=
```
n = 4; (* integer ≥ 0 *)

REF = Log[2^(-n (n-1)) Product[ν^(ν-2 n+2) (ν + α)^(ν-1) (ν + β)^(ν-1) (ν + n + α + β)^(n-ν), {ν, 1, n}]];

RES = - n (n - 1) Log[2] + FracA[n] + FracB[n, α] + FracB[n, β] + FracC[n, α + β];

REF == RES // FullSimplify[#, Assumptions → {α > -1, β > -1}] &

Clear[n, f];
```

Out[ ]= True

In[ ]:=
```
n = 2; (* integer ≥ 0 *)

REF = - n (n - 1) Log[2] + FracA[n] + FracB[n, α] + FracB[n, β] + FracC[n, α + β];

RES = - n (n - 1) Log[2] + FracA2[n] + FracB2[n, α] + FracB2[n, β] + FracC2[n, α + β];

f = REF - RES;

ContourPlot[f, {α, -1, 3}, {β, -1, 3}, WorkingPrecision → 64]

Clear[n, f];
```

Out[ ]=



In[ ]:=
```
acc = 64;

n = 2; (* integer ≥ 0 *)
α = N[Sqrt[ⅇ], acc];
β = N[1 / π, acc];

REF = - n (n - 1) Log[2] + FracA[n] + FracB[n, α] + FracB[n, β] + FracC[n, α + β];

RES = - n (n - 1) Log[2] + FracA2[n] + FracB2[n, α] + FracB2[n, β] + FracC2[n, α + β];

N[REF - RES, acc]

Clear[α, β, n, f];
```

Out[ ]= $0. \times 10^{-61}$

Asymptotics : $\mathfrak{A}_n$

Application of LogGamma and Zeta prime asymptotics

```
Zeta^(1,0)[-1, n + 1] - Zeta'[-1] - 2 (n - 1) Log[Gamma[1 + n]]; (* for comparison *)
```

In[ ]:=
```
TMP01 =
  AsymptoticsHurwitzZetaPrime[1, n] - Zeta'[-1] - 2 (n - 1) AsymptoticsLogGamma[1, n] // Expand
```

Out[ ]= $$-2\,n + \frac{7\,n^2}{4} + \mathrm{Log}[\mathrm{Glaisher}] + \frac{13\,\mathrm{Log}[n]}{12} + \frac{3}{2}\,n\,\mathrm{Log}[n] - \frac{3}{2}\,n^2\,\mathrm{Log}[n] + \mathrm{Log}[2\,\pi] - n\,\mathrm{Log}[2\,\pi] + \mathrm{PoincareSum}\left[\frac{(-1)^m\,n^{-m}\,\mathrm{Zeta}[-1-m]}{m\,(1+m)}, \{m, 1, \infty\}\right] - 2\,\mathrm{PoincareSum}\left[\frac{(-1)^{-1+m}\,n^{-m}\,\mathrm{Zeta}[-m]}{m}, \{m, 1, \infty\}\right] + 2\,n\,\mathrm{PoincareSum}\left[\frac{(-1)^{-1+m}\,n^{-m}\,\mathrm{Zeta}[-m]}{m}, \{m, 1, \infty\}\right]$$

Simplification

In[ ]:=
```
TMP01 /. PoincareSumFactorUnderSum
```

Out[ ]= $$-2\,n + \frac{7\,n^2}{4} + \mathrm{Log}[\mathrm{Glaisher}] + \frac{13\,\mathrm{Log}[n]}{12} + \frac{3}{2}\,n\,\mathrm{Log}[n] - \frac{3}{2}\,n^2\,\mathrm{Log}[n] + \mathrm{Log}[2\,\pi] - n\,\mathrm{Log}[2\,\pi] + \mathrm{PoincareSum}\left[\frac{(-1)^m\,n^{-m}\,\mathrm{Zeta}[-1-m]}{m\,(1+m)}, \{m, 1, \infty\}\right] + \mathrm{PoincareSum}\left[\frac{2\,(-1)^{-1+m}\,n^{1-m}\,\mathrm{Zeta}[-m]}{m}, \{m, 1, \infty\}\right] + \mathrm{PoincareSum}\left[-\frac{2\,(-1)^{-1+m}\,n^{-m}\,\mathrm{Zeta}[-m]}{m}, \{m, 1, \infty\}\right]$$

In[ ]:= $-2\,n+\frac{7\,n^2}{4}+\text{Log[Glaisher]}+\frac{13\,\text{Log}[n]}{12}+\frac{3}{2}\,n\,\text{Log}[n]-\frac{3}{2}\,n^2\,\text{Log}[n]+\text{Log}[2\,\pi]-n\,\text{Log}[2\,\pi]+\text{PoincareSum}\left[\frac{(-1)^m\,n^{-m}\,\text{Zeta}[-1-m]}{m\,(1+m)},\{m,1,\infty\}\right]+\left(\text{PoincareSum}\left[\frac{2\,(-1)^{-1+m}\,n^{1-m}\,\text{Zeta}[-m]}{m},\{m,1,\infty\}\right]\text{ /. PoincareSumIndexShiftUp[1]}\right)+\text{PoincareSum}\left[-\frac{2\,(-1)^{-1+m}\,n^{-m}\,\text{Zeta}[-m]}{m},\{m,1,\infty\}\right]$

Out[ ]= $-2\,n+\frac{7\,n^2}{4}+\text{Log[Glaisher]}+\frac{13\,\text{Log}[n]}{12}+\frac{3}{2}\,n\,\text{Log}[n]-\frac{3}{2}\,n^2\,\text{Log}[n]+\text{Log}[2\,\pi]-n\,\text{Log}[2\,\pi]+\text{PoincareSum}\left[\frac{2\,(-1)^m\,n^{-m}\,\text{Zeta}[-1-m]}{1+m},\{m,0,\infty\}\right]+\text{PoincareSum}\left[\frac{(-1)^m\,n^{-m}\,\text{Zeta}[-1-m]}{m\,(1+m)},\{m,1,\infty\}\right]+\text{PoincareSum}\left[-\frac{2\,(-1)^{-1+m}\,n^{-m}\,\text{Zeta}[-m]}{m},\{m,1,\infty\}\right]$

In[ ]:= $\text{PoincareSum}\left[\frac{2\,(-1)^m\,n^{-m}\,\text{Zeta}[-1-m]}{1+m},\{m,0,\infty\}\right]\text{ /. PoincareSumSplitOffTerms[1]}$

Out[ ]= $-\frac{1}{6}+\text{PoincareSum}\left[\frac{2\,(-1)^m\,n^{-m}\,\text{Zeta}[-1-m]}{1+m},\{m,1,\infty\}\right]$

In[ ]:= $\text{TMP02}=-2\,n+\frac{7\,n^2}{4}+\text{Log[Glaisher]}+\frac{13\,\text{Log}[n]}{12}+\frac{3}{2}\,n\,\text{Log}[n]-\frac{3}{2}\,n^2\,\text{Log}[n]+\text{Log}[2\,\pi]-n\,\text{Log}[2\,\pi]+\left(\text{PoincareSum}\left[\frac{2\,(-1)^m\,n^{-m}\,\text{Zeta}[-1-m]}{1+m},\{m,0,\infty\}\right]\text{ /. PoincareSumSplitOffTerms[1]}\right)+\text{PoincareSum}\left[\frac{(-1)^m\,n^{-m}\,\text{Zeta}[-1-m]}{m\,(1+m)},\{m,1,\infty\}\right]+\text{PoincareSum}\left[-\frac{2\,(-1)^{-1+m}\,n^{-m}\,\text{Zeta}[-m]}{m},\{m,1,\infty\}\right]$

Out[ ]= $-\frac{1}{6}-2\,n+\frac{7\,n^2}{4}+\text{Log[Glaisher]}+\frac{13\,\text{Log}[n]}{12}+\frac{3}{2}\,n\,\text{Log}[n]-\frac{3}{2}\,n^2\,\text{Log}[n]+\text{Log}[2\,\pi]-n\,\text{Log}[2\,\pi]+\text{PoincareSum}\left[\frac{2\,(-1)^m\,n^{-m}\,\text{Zeta}[-1-m]}{1+m},\{m,1,\infty\}\right]+\text{PoincareSum}\left[\frac{(-1)^m\,n^{-m}\,\text{Zeta}[-1-m]}{m\,(1+m)},\{m,1,\infty\}\right]+\text{PoincareSum}\left[-\frac{2\,(-1)^{-1+m}\,n^{-m}\,\text{Zeta}[-m]}{m},\{m,1,\infty\}\right]$

In[ ]:= TMP02 //. PoincareSumCollect

Out[ ]= $-\frac{1}{6}-2n+\frac{7n^2}{4}+\text{Log[Glaisher]}+\frac{13\,\text{Log}[n]}{12}+\frac{3}{2}n\,\text{Log}[n]-\frac{3}{2}n^2\,\text{Log}[n]+\text{Log}[2\pi]-n\,\text{Log}[2\pi]+\text{PoincareSum}\left[\frac{2(-1)^m n^{-m}\,\text{Zeta}[-1-m]}{1+m}+\frac{(-1)^m n^{-m}\,\text{Zeta}[-1-m]}{m(1+m)}-\frac{2(-1)^{-1+m}n^{-m}\,\text{Zeta}[-m]}{m},\{m,1,\infty\}\right]$

In[ ]:= $\text{TMP}=\frac{2(-1)^m n^{-m}\,\text{Zeta}[-1-m]}{1+m}+\frac{(-1)^m n^{-m}\,\text{Zeta}[-1-m]}{m(1+m)}-\frac{2(-1)^{-1+m}n^{-m}\,\text{Zeta}[-m]}{m}$ // Simplify

Out[ ]= $\frac{(-1)^m n^{-m}((1+2m)\,\text{Zeta}[-1-m]+2(1+m)\,\text{Zeta}[-m])}{m(1+m)}$

In[ ]:= $\frac{((1+2m)\,\text{Zeta}[-1-m]+2(1+m)\,\text{Zeta}[-m])}{(1+m)}$ // Apart

Out[ ]= $\frac{(1+2m)\,\text{Zeta}[-1-m]}{1+m}+2\,\text{Zeta}[-m]$

In[ ]:= $\text{TMP}==\frac{(-1)^m}{m}\left(2\,\text{Zeta}[-m]+\frac{2m+1}{m+1}\,\text{Zeta}[-1-m]\right)n^{-m}$ // FullSimplify

Out[ ]= True

In[ ]:= $\text{TMP03}=-\frac{1}{6}-2n+\frac{7n^2}{4}+\text{Log[Glaisher]}+\frac{13\,\text{Log}[n]}{12}+\frac{3}{2}n\,\text{Log}[n]-\frac{3}{2}n^2\,\text{Log}[n]+\text{Log}[2\pi]-n\,\text{Log}[2\pi]+\text{PoincareSum}\left[\frac{(-1)^m}{m}\left(2\,\text{Zeta}[-m]+\frac{2m+1}{m+1}\,\text{Zeta}[-1-m]\right)n^{-m},\{m,1,\infty\}\right];$

In[ ]:= $\text{TMP}=-\frac{1}{6}-2n+\frac{7n^2}{4}+\text{Log[Glaisher]}+\frac{13\,\text{Log}[n]}{12}+\frac{3}{2}n\,\text{Log}[n]-\frac{3}{2}n^2\,\text{Log}[n]+\text{Log}[2\pi]-n\,\text{Log}[2\pi];$

$\text{TMP}==-\frac{3}{2}n^2\,\text{Log}[n]+\frac{7}{4}n^2+\frac{3}{2}n\,\text{Log}[n]-(2+\text{Log}[2\pi])\,n+\frac{13}{12}\text{Log}[n]+\text{Log[Glaisher]}-\frac{1}{6}+\text{Log}[2\pi]$ // FullSimplify

Out[ ]= True

In[ ]:= $\text{TMP04}=-\frac{3}{2}n^2\,\text{Log}[n]+\frac{7}{4}n^2+\frac{3}{2}n\,\text{Log}[n]-(2+\text{Log}[2\pi])\,n+\frac{13}{12}\text{Log}[n]+\text{Log[Glaisher]}-\frac{1}{6}+\text{Log}[2\pi]+\text{PoincareSum}\left[\frac{(-1)^m}{m}\left(2\,\text{Zeta}[-m]+\frac{2m+1}{m+1}\,\text{Zeta}[-1-m]\right)n^{-m},\{m,1,\infty\}\right];$

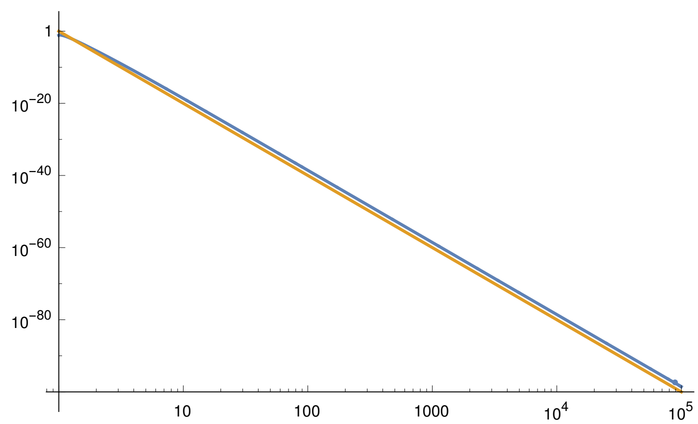
Formula

In[ ]:=
```
ASYMPFracA[n_] := -3/2 n^2 Log[n] + 7/4 n^2 + 3/2 n Log[n] - (2 + Log[2 π]) n + 13/12 Log[n] + Log[Glaisher] -
   1/6 + Log[2 π] + PoincareSum[(-1)^m/m (2 Zeta[-m] + (2 m + 1)/(m + 1) Zeta[-1 - m]) n^-m, {m, 1, ∞}];
```

Cross Check

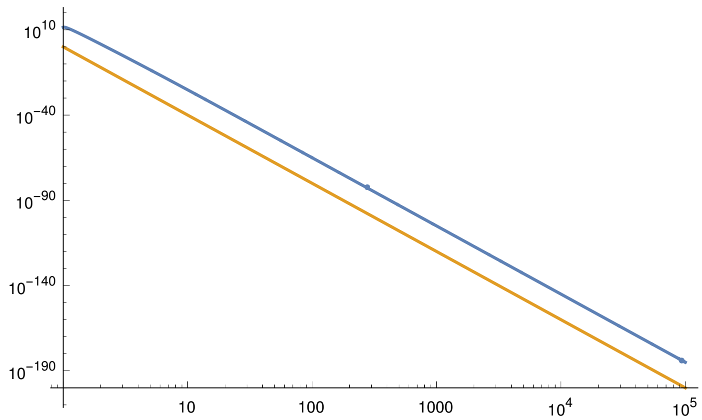
In[ ]:=
```
REF = Zeta^(1,0)[-1, n + 1] - Zeta'[-1] - 2 (n - 1) Log[Gamma[1 + n]];

K = 20;
α =.;
β =.;

ASYMP = ASYMPFracA[n] /. PoincareSumNormalize[K - 1];

LogLogPlot[{Abs[REF - ASYMP], n^-K}, {n, 1, 100 000}, WorkingPrecision → 256]

Clear[α, β, K];
```

Out[ ]=



In[ ]:=
```
REF = Zeta^(1,0)[-1, n + 1] - Zeta'[-1] - 2 (n - 1) Log[Gamma[1 + n]];

K = 40;
α =.;
β =.;

ASYMP = ASYMPFracA[n] /. PoincareSumNormalize[K - 1];

LogLogPlot[{Abs[REF - ASYMP], n^-K}, {n, 1, 100 000}, WorkingPrecision → 256]

Clear[α, β, K];
```

Out[ ]=



Asymptotics : $\mathcal{B}_n(\alpha)$

Application of LogGamma and Zeta prime asymptotics

```
Zeta^(1,0)[-1, n + α + 1] - Zeta^(1,0)[-1, α + 1] - (α + 1) Log[Pochhammer[α + 1, n]];
(* for comparison *)

Zeta^(1,0)[-1, n + α + 1] - Zeta^(1,0)[-1, α + 1] - (α + 1) Log[Gamma[n + α + 1]] + (α + 1) Log[Gamma[α + 1]];
```

In[ ]:= **TMP01 = AsymptoticsHurwitzZetaPrime[α + 1, n] − Zeta^(1,0)[−1, α + 1] − (α + 1) AsymptoticsLogGamma[α + 1, n] + (α + 1) Log[Gamma[α + 1]] // Expand**

Out[ ]= $n - \frac{n^2}{4} + n\,\alpha - \text{HurwitzZeta}[-1, 1+\alpha] - \frac{\text{Log}[n]}{2} - n\,\text{Log}[n] + \frac{1}{2} n^2\,\text{Log}[n] - \frac{3}{2}\alpha\,\text{Log}[n] - n\,\alpha\,\text{Log}[n] - \alpha^2\,\text{Log}[n] - \text{HurwitzZeta}[-1, 1+\alpha]\,\text{Log}[n] - n\,\text{HurwitzZeta}[0, 1+\alpha]\,\text{Log}[n] - \frac{1}{2}\text{Log}[2\pi] - \frac{1}{2}\alpha\,\text{Log}[2\pi] + \text{Log}[\text{Gamma}[1+\alpha]] + \alpha\,\text{Log}[\text{Gamma}[1+\alpha]] + \text{PoincareSum}\left[\frac{(-1)^m n^{-m}\,\text{HurwitzZeta}[-1-m, 1+\alpha]}{m\,(1+m)}, \{m, 1, \infty\}\right] + \text{PoincareSum}\left[\frac{(-1)^{-1+m} n^{-m}\,\text{HurwitzZeta}[-m, 1+\alpha]}{m}, \{m, 1, \infty\}\right] + \alpha\,\text{PoincareSum}\left[\frac{(-1)^{-1+m} n^{-m}\,\text{HurwitzZeta}[-m, 1+\alpha]}{m}, \{m, 1, \infty\}\right] - \text{Zeta}^{(1,0)}[-1, 1+\alpha]$

Simplification

In[ ]:= **TMP02 = TMP01 /. PoincareSumFactorUnderSum**

Out[ ]= $n - \frac{n^2}{4} + n\,\alpha - \text{HurwitzZeta}[-1, 1+\alpha] - \frac{\text{Log}[n]}{2} - n\,\text{Log}[n] + \frac{1}{2} n^2\,\text{Log}[n] - \frac{3}{2}\alpha\,\text{Log}[n] - n\,\alpha\,\text{Log}[n] - \alpha^2\,\text{Log}[n] - \text{HurwitzZeta}[-1, 1+\alpha]\,\text{Log}[n] - n\,\text{HurwitzZeta}[0, 1+\alpha]\,\text{Log}[n] - \frac{1}{2}\text{Log}[2\pi] - \frac{1}{2}\alpha\,\text{Log}[2\pi] + \text{Log}[\text{Gamma}[1+\alpha]] + \alpha\,\text{Log}[\text{Gamma}[1+\alpha]] + \text{PoincareSum}\left[\frac{(-1)^m n^{-m}\,\text{HurwitzZeta}[-1-m, 1+\alpha]}{m\,(1+m)}, \{m, 1, \infty\}\right] + \text{PoincareSum}\left[\frac{(-1)^{-1+m} n^{-m}\,\text{HurwitzZeta}[-m, 1+\alpha]}{m}, \{m, 1, \infty\}\right] + \text{PoincareSum}\left[\frac{(-1)^{-1+m} n^{-m}\,\alpha\,\text{HurwitzZeta}[-m, 1+\alpha]}{m}, \{m, 1, \infty\}\right] - \text{Zeta}^{(1,0)}[-1, 1+\alpha]$

In[ ]:= TMP02 //. PoincareSumCollect

Out[ ]= $n - \frac{n^2}{4} + n\,\alpha - \text{HurwitzZeta}[-1, 1+\alpha] - \frac{\text{Log}[n]}{2} - n\,\text{Log}[n] + \frac{1}{2} n^2\,\text{Log}[n] - \frac{3}{2}\alpha\,\text{Log}[n] - n\,\alpha\,\text{Log}[n] - \alpha^2\,\text{Log}[n] - \text{HurwitzZeta}[-1, 1+\alpha]\,\text{Log}[n] - n\,\text{HurwitzZeta}[0, 1+\alpha]\,\text{Log}[n] - \frac{1}{2}\text{Log}[2\pi] - \frac{1}{2}\alpha\,\text{Log}[2\pi] + \text{Log}[\text{Gamma}[1+\alpha]] + \alpha\,\text{Log}[\text{Gamma}[1+\alpha]] + \text{PoincareSum}\left[\frac{(-1)^m n^{-m}\,\text{HurwitzZeta}[-1-m, 1+\alpha]}{m(1+m)} + \frac{(-1)^{-1+m} n^{-m}\,\text{HurwitzZeta}[-m, 1+\alpha]}{m} + \frac{(-1)^{-1+m} n^{-m}\,\alpha\,\text{HurwitzZeta}[-m, 1+\alpha]}{m}, \{m, 1, \infty\}\right] - \text{Zeta}^{(1,0)}[-1, 1+\alpha]$

In[ ]:= $\text{TMP} = \frac{(-1)^m n^{-m}\,\text{HurwitzZeta}[-1-m, 1+\alpha]}{m(1+m)} + \frac{(-1)^{-1+m} n^{-m}\,\text{HurwitzZeta}[-m, 1+\alpha]}{m} + \frac{(-1)^{-1+m} n^{-m}\,\alpha\,\text{HurwitzZeta}[-m, 1+\alpha]}{m}$ // Simplify

Out[ ]= $\frac{(-1)^{1+m} n^{-m}\left(-\text{HurwitzZeta}[-1-m, 1+\alpha] + (1+m)(1+\alpha)\,\text{HurwitzZeta}[-m, 1+\alpha]\right)}{m(1+m)}$

In[ ]:= $\text{TMP} == \frac{(-1)^{m-1}}{m}\left((1+\alpha)\,\text{HurwitzZeta}[-m, 1+\alpha] - \frac{\text{HurwitzZeta}[-1-m, 1+\alpha]}{1+m}\right) n^{-m}$ // Simplify

Out[ ]= True

$\text{TMP03} = n - \frac{n^2}{4} + n\,\alpha - \text{HurwitzZeta}[-1, 1+\alpha] - \frac{\text{Log}[n]}{2} - n\,\text{Log}[n] + \frac{1}{2} n^2\,\text{Log}[n] - \frac{3}{2}\alpha\,\text{Log}[n] - n\,\alpha\,\text{Log}[n] - \alpha^2\,\text{Log}[n] - \text{HurwitzZeta}[-1, 1+\alpha]\,\text{Log}[n] - n\,\text{HurwitzZeta}[0, 1+\alpha]\,\text{Log}[n] - \frac{1}{2}\text{Log}[2\pi] - \frac{1}{2}\alpha\,\text{Log}[2\pi] + \text{Log}[\text{Gamma}[1+\alpha]] + \alpha\,\text{Log}[\text{Gamma}[1+\alpha]] - \text{Zeta}^{(1,0)}[-1, 1+\alpha] + \text{PoincareSum}\left[\frac{(-1)^{m-1}}{m}\left((1+\alpha)\,\text{HurwitzZeta}[-m, 1+\alpha] - \frac{\text{HurwitzZeta}[-1-m, 1+\alpha]}{1+m}\right) n^{-m}, \{m, 1, \infty\}\right];$

In[ ]:=
```
TMP = n - n^2/4 + n α - HurwitzZeta[-1, 1 + α] - Log[n]/2 - n Log[n] + 1/2 n^2 Log[n] - 3/2 α Log[n] -
  n α Log[n] - α^2 Log[n] - HurwitzZeta[-1, 1 + α] Log[n] - n HurwitzZeta[0, 1 + α] Log[n] -
  1/2 Log[2 π] - 1/2 α Log[2 π] + Log[Gamma[1 + α]] + α Log[Gamma[1 + α]] - Zeta^(1,0)[-1, 1 + α];
```

Auxiliary computations

In[ ]:=
```
HurwitzZeta[0, 1 + α] // FunctionExpand
```

Out[ ]= $-\frac{1}{2} - \alpha$

In[ ]:=
```
-n Log[n] - n α Log[n] - n HurwitzZeta[0, 1 + α] Log[n] // FullSimplify
```

Out[ ]= $-\frac{1}{2}\, n\, \mathrm{Log}[n]$

In[ ]:=
```
-Log[n]/2 - 3/2 α Log[n] - α^2 Log[n] - HurwitzZeta[-1, 1 + α] Log[n] // FullSimplify
```

Out[ ]= $-\frac{1}{12} (5 + 6\, \alpha\, (2 + \alpha))\, \mathrm{Log}[n]$

In[ ]:=
```
-1/12 (5 + 6 α (2 + α)) == 1/2 (1/6 - (α + 1)^2) // FullSimplify
```

Out[ ]= True

Leading terms

In[ ]:=
```
TMP == 1/2 n^2 Log[n] - 1/4 n^2 - 1/2 n Log[n] + (α + 1) n + 1/2 (1/6 - (α + 1)^2) Log[n] - HurwitzZeta[-1, 1 + α] -
  1/2 Log[2 π] - 1/2 α Log[2 π] + (α + 1) Log[Gamma[α + 1]] - Zeta^(1,0)[-1, 1 + α] // FullSimplify
```

Out[ ]= True

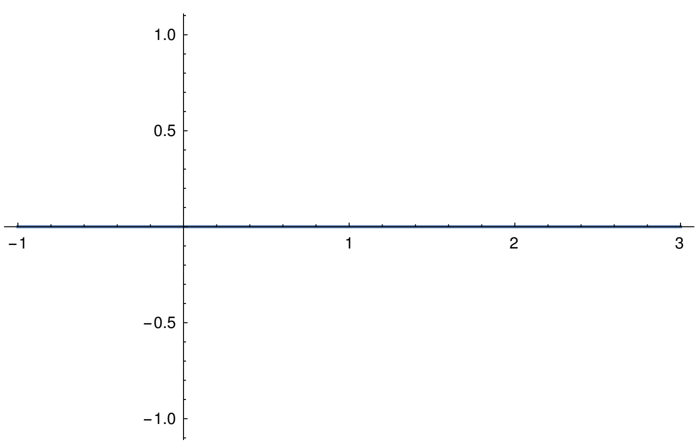
of which is the constant term

In[ ]:=
```
-HurwitzZeta[-1, 1 + α] // FunctionExpand
```

Out[ ]= $\frac{1}{2}\left(-\frac{5}{6} - \alpha + (1 + \alpha)^2\right)$

In[ ]:=
```
REF = -HurwitzZeta[-1, 1 + α] - 1/2 Log[2 π] -
   1/2 α Log[2 π] + (α + 1) Log[Gamma[α + 1]] - Zeta^(1,0)[-1, 1 + α];

RES = Log[Glaisher] - PolyGamma[-2, α + 1] + (α + 1) Log[Gamma[α + 1]];

f = RES - REF // FunctionExpand // FullSimplify

Plot[f , {α, -1, 3}, WorkingPrecision → 64]

RES1 = RES /. PolyGamma[-2, x_] → (1 - x) x/2 + x/2 Log[2 π] - Zeta'[-1] + Zeta^(1,0)[-1, x];

RES1 == REF // FullSimplify

Clear[f];
```

Out[ ]= $-\frac{1}{12}-\frac{\alpha}{2}-\frac{\alpha^2}{2}+\text{Log[Glaisher]}+\frac{1}{2}(1+\alpha)\,\text{Log}[2\,\pi]-\text{PolyGamma}[-2,\,1+\alpha]+\text{Zeta}^{(1,0)}[-1,\,1+\alpha]$

Out[ ]=


Out[ ]= True

In[ ]:=
```
TMP == 1/2 n^2 Log[n] - 1/4 n^2 - 1/2 n Log[n] + (α + 1) n + 1/2 (1/6 - (α + 1)^2) Log[n] +
    Log[Glaisher] - PolyGamma[-2, α + 1] + (α + 1) Log[Gamma[α + 1]];
% /. PolyGamma[-2, x_] → ((1 - x) x)/2 + x/2 Log[2 π] - Zeta'[-1] + Zeta^(1,0)[-1, x];
% // FullSimplify
```

Out[ ]=
```
True
```

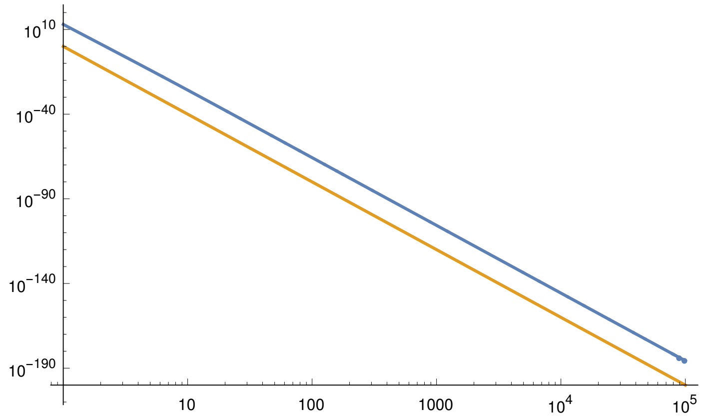
In[ ]:=
```
TMP04 = 1/2 n^2 Log[n] - 1/4 n^2 - 1/2 n Log[n] + (α + 1) n + 1/2 (1/6 - (α + 1)^2) Log[n] +
    Log[Glaisher] - PolyGamma[-2, α + 1] + (α + 1) Log[Gamma[α + 1]] + PoincareSum[
     (-1)^(m-1)/m ((α + 1) HurwitzZeta[-m, α + 1] - HurwitzZeta[-1 - m, α + 1]/(1 + m)) n^-m, {m, 1, ∞}];
```

Formula

In[ ]:=
```
ASYMPFracB[n_, α_] := 1/2 n^2 Log[n] - 1/4 n^2 - 1/2 n Log[n] + (α + 1) n + 1/2 (1/6 - (α + 1)^2) Log[n] +
    Log[Glaisher] - PolyGamma[-2, α + 1] + (α + 1) Log[Gamma[α + 1]] + PoincareSum[
     (-1)^(m-1)/m ((α + 1) HurwitzZeta[-m, α + 1] - HurwitzZeta[-1 - m, α + 1]/(1 + m)) n^-m, {m, 1, ∞}];
```

Cross Check

In[ ]:=
```
REF = Zeta^(1,0)[-1, n + α + 1] - Zeta^(1,0)[-1, α + 1] - (α + 1) Log[Pochhammer[α + 1, n]];

K = 40;
α = ⅇ - 1;
β =.;

ASYMP = ASYMPFracB[n, α] /. PoincareSumNormalize[K - 1];

LogLogPlot[{Abs[REF - ASYMP], n^-K}, {n, 1, 100 000}, WorkingPrecision → 256]

Clear[α, β, K];
```

Out[ ]=


Asymptotics : C_n (b)

Application of LogGamma and Zeta prime asymptotics

```
(2 n + b) Log[Pochhammer[n + b + 1, n]] - Zeta^(1,0)[-1, 2 n + b + 1] + Zeta^(1,0)[-1, n + b + 1];
(* for comparison *)

(2 n + b) Log[Gamma[2 n + b + 1]] - (2 n + b) Log[Gamma[n + b + 1]] -
   Zeta^(1,0)[-1, 2 n + b + 1] + Zeta^(1,0)[-1, n + b + 1];
```

In[ ]:= **TMP01 = (2 n + b) AsymptoticsLogGamma[b + 1, 2 n] - (2 n + b) AsymptoticsLogGamma[b + 1, n] - AsymptoticsHurwitzZetaPrime[b + 1, 2 n] + AsymptoticsHurwitzZetaPrime[b + 1, n] // Expand**

Out[ ]= $-b\,n-\frac{5\,n^2}{4}-\frac{1}{2}\,b\,\text{Log}[n]-b^2\,\text{Log}[n]-n\,\text{Log}[n]-3\,b\,n\,\text{Log}[n]-\frac{3}{2}\,n^2\,\text{Log}[n]-$
$\text{HurwitzZeta}[-1,\,1+b]\,\text{Log}[n]-n\,\text{HurwitzZeta}[0,\,1+b]\,\text{Log}[n]+\frac{1}{2}\,b\,\text{Log}[2\,n]+$
$b^2\,\text{Log}[2\,n]+n\,\text{Log}[2\,n]+4\,b\,n\,\text{Log}[2\,n]+2\,n^2\,\text{Log}[2\,n]+\text{HurwitzZeta}[-1,\,1+b]\,\text{Log}[2\,n]+$
$2\,n\,\text{HurwitzZeta}[0,\,1+b]\,\text{Log}[2\,n]+\text{PoincareSum}\left[\frac{(-1)^m\,n^{-m}\,\text{HurwitzZeta}[-1-m,\,1+b]}{m\,(1+m)},\,\{m,\,1,\,\infty\}\right]-$
$\text{PoincareSum}\left[\frac{\left(-\frac{1}{2}\right)^m\,n^{-m}\,\text{HurwitzZeta}[-1-m,\,1+b]}{m\,(1+m)},\,\{m,\,1,\,\infty\}\right]+$
$b\,\text{PoincareSum}\left[\frac{(-1)^{-1+m}\,n^{-m}\,\text{HurwitzZeta}[-m,\,1+b]}{m},\,\{m,\,1,\,\infty\}\right]+$
$2\,n\,\text{PoincareSum}\left[\frac{(-1)^{-1+m}\,n^{-m}\,\text{HurwitzZeta}[-m,\,1+b]}{m},\,\{m,\,1,\,\infty\}\right]-$
$b\,\text{PoincareSum}\left[\frac{(-1)^{-1+m}\,2^{-m}\,n^{-m}\,\text{HurwitzZeta}[-m,\,1+b]}{m},\,\{m,\,1,\,\infty\}\right]-$
$2\,n\,\text{PoincareSum}\left[\frac{(-1)^{-1+m}\,2^{-m}\,n^{-m}\,\text{HurwitzZeta}[-m,\,1+b]}{m},\,\{m,\,1,\,\infty\}\right]$

Simplification

In[ ]:= `TMP02 = TMP01 /. PoincareSumFactorUnderSum`

Out[ ]=
$$-b\,n-\frac{5\,n^2}{4}-\frac{1}{2}\,b\,\text{Log}[n]-b^2\,\text{Log}[n]-n\,\text{Log}[n]-3\,b\,n\,\text{Log}[n]-\frac{3}{2}\,n^2\,\text{Log}[n]-\text{HurwitzZeta}[-1,\,1+b]\,\text{Log}[n]-n\,\text{HurwitzZeta}[0,\,1+b]\,\text{Log}[n]+\frac{1}{2}\,b\,\text{Log}[2\,n]+b^2\,\text{Log}[2\,n]+n\,\text{Log}[2\,n]+4\,b\,n\,\text{Log}[2\,n]+2\,n^2\,\text{Log}[2\,n]+\text{HurwitzZeta}[-1,\,1+b]\,\text{Log}[2\,n]+2\,n\,\text{HurwitzZeta}[0,\,1+b]\,\text{Log}[2\,n]+\text{PoincareSum}\left[\frac{(-1)^m\,n^{-m}\,\text{HurwitzZeta}[-1-m,\,1+b]}{m\,(1+m)},\,\{m,\,1,\,\infty\}\right]+\text{PoincareSum}\left[\frac{(-1)^{1+m}\,2^{-m}\,n^{-m}\,\text{HurwitzZeta}[-1-m,\,1+b]}{m\,(1+m)},\,\{m,\,1,\,\infty\}\right]+\text{PoincareSum}\left[\frac{2\,(-1)^{-1+m}\,n^{1-m}\,\text{HurwitzZeta}[-m,\,1+b]}{m},\,\{m,\,1,\,\infty\}\right]+\text{PoincareSum}\left[\frac{(-1)^m\,2^{1-m}\,n^{1-m}\,\text{HurwitzZeta}[-m,\,1+b]}{m},\,\{m,\,1,\,\infty\}\right]+\text{PoincareSum}\left[\frac{(-1)^{-1+m}\,b\,n^{-m}\,\text{HurwitzZeta}[-m,\,1+b]}{m},\,\{m,\,1,\,\infty\}\right]+\text{PoincareSum}\left[\frac{\left(-\frac{1}{2}\right)^m\,b\,n^{-m}\,\text{HurwitzZeta}[-m,\,1+b]}{m},\,\{m,\,1,\,\infty\}\right]$$

In[ ]:= TMP03 = -b n - $\frac{5 n^2}{4}$ - $\frac{1}{2}$ b Log[n] - b^2 Log[n] - n Log[n] - 3 b n Log[n] -
$\frac{3}{2}$ n^2 Log[n] - HurwitzZeta[-1, 1 + b] Log[n] - n HurwitzZeta[0, 1 + b] Log[n] +
$\frac{1}{2}$ b Log[2 n] + b^2 Log[2 n] + n Log[2 n] + 4 b n Log[2 n] + 2 n^2 Log[2 n] +
HurwitzZeta[-1, 1 + b] Log[2 n] + 2 n HurwitzZeta[0, 1 + b] Log[2 n] +
PoincareSum[$\frac{(-1)^m n^{-m} \text{HurwitzZeta}[-1-m, 1+b]}{m (1+m)}$, {m, 1, ∞}] +
PoincareSum[$\frac{(-1)^{1+m} 2^{-m} n^{-m} \text{HurwitzZeta}[-1-m, 1+b]}{m (1+m)}$, {m, 1, ∞}] +
(PoincareSum[$\frac{2 (-1)^{-1+m} n^{1-m} \text{HurwitzZeta}[-m, 1+b]}{m}$, {m, 1, ∞}] + PoincareSum[
$\frac{(-1)^m 2^{1-m} n^{1-m} \text{HurwitzZeta}[-m, 1+b]}{m}$, {m, 1, ∞}] /. PoincareSumIndexShiftUp[1]) +
PoincareSum[$\frac{(-1)^{-1+m} b n^{-m} \text{HurwitzZeta}[-m, 1+b]}{m}$, {m, 1, ∞}] +
PoincareSum[$\frac{\left(-\frac{1}{2}\right)^m b n^{-m} \text{HurwitzZeta}[-m, 1+b]}{m}$, {m, 1, ∞}]

Out[ ]= $$-b\,n-\frac{5\,n^2}{4}-\frac{1}{2}\,b\,\text{Log}[n]-b^2\,\text{Log}[n]-n\,\text{Log}[n]-3\,b\,n\,\text{Log}[n]-\frac{3}{2}\,n^2\,\text{Log}[n]-$$
$$\text{HurwitzZeta}[-1,\,1+b]\,\text{Log}[n]-n\,\text{HurwitzZeta}[0,\,1+b]\,\text{Log}[n]+\frac{1}{2}\,b\,\text{Log}[2\,n]+$$
$$b^2\,\text{Log}[2\,n]+n\,\text{Log}[2\,n]+4\,b\,n\,\text{Log}[2\,n]+2\,n^2\,\text{Log}[2\,n]+\text{HurwitzZeta}[-1,\,1+b]\,\text{Log}[2\,n]+$$
$$2\,n\,\text{HurwitzZeta}[0,\,1+b]\,\text{Log}[2\,n]+\text{PoincareSum}\Big[\frac{2\,(-1)^m\,n^{-m}\,\text{HurwitzZeta}[-1-m,\,1+b]}{1+m},\,\{m,\,0,\,\infty\}\Big]+$$
$$\text{PoincareSum}\Big[\frac{(-1)^{1+m}\,2^{-m}\,n^{-m}\,\text{HurwitzZeta}[-1-m,\,1+b]}{1+m},\,\{m,\,0,\,\infty\}\Big]+$$
$$\text{PoincareSum}\Big[\frac{(-1)^m\,n^{-m}\,\text{HurwitzZeta}[-1-m,\,1+b]}{m\,(1+m)},\,\{m,\,1,\,\infty\}\Big]+$$
$$\text{PoincareSum}\Big[\frac{(-1)^{1+m}\,2^{-m}\,n^{-m}\,\text{HurwitzZeta}[-1-m,\,1+b]}{m\,(1+m)},\,\{m,\,1,\,\infty\}\Big]+$$
$$\text{PoincareSum}\Big[\frac{(-1)^{-1+m}\,b\,n^{-m}\,\text{HurwitzZeta}[-m,\,1+b]}{m},\,\{m,\,1,\,\infty\}\Big]+$$
$$\text{PoincareSum}\Big[\frac{\left(-\frac{1}{2}\right)^m\,b\,n^{-m}\,\text{HurwitzZeta}[-m,\,1+b]}{m},\,\{m,\,1,\,\infty\}\Big]$$

In[ ]:=
```
TMP04 = -b n - (5 n^2)/4 - 1/2 b Log[n] - b^2 Log[n] - n Log[n] - 3 b n Log[n] -
   3/2 n^2 Log[n] - HurwitzZeta[-1, 1 + b] Log[n] - n HurwitzZeta[0, 1 + b] Log[n] +
   1/2 b Log[2 n] + b^2 Log[2 n] + n Log[2 n] + 4 b n Log[2 n] + 2 n^2 Log[2 n] +
   HurwitzZeta[-1, 1 + b] Log[2 n] + 2 n HurwitzZeta[0, 1 + b] Log[2 n] +
   (PoincareSum[(2 (-1)^m n^-m HurwitzZeta[-1 - m, 1 + b])/(1 + m), {m, 0, ∞}] + PoincareSum[
       ((-1)^(1+m) 2^-m n^-m HurwitzZeta[-1 - m, 1 + b])/(1 + m), {m, 0, ∞}] /. PoincareSumSplitOffTerms[1]) +
   PoincareSum[((-1)^m n^-m HurwitzZeta[-1 - m, 1 + b])/(m (1 + m)), {m, 1, ∞}] +
   PoincareSum[((-1)^(1+m) 2^-m n^-m HurwitzZeta[-1 - m, 1 + b])/(m (1 + m)), {m, 1, ∞}] +
   PoincareSum[((-1)^(-1+m) b n^-m HurwitzZeta[-m, 1 + b])/m, {m, 1, ∞}] +
   PoincareSum[((-1/2)^m b n^-m HurwitzZeta[-m, 1 + b])/m, {m, 1, ∞}]
```

Out[ ]= $-b\,n-\frac{5\,n^2}{4}+\text{HurwitzZeta}[-1,\,1+b]-\frac{1}{2}\,b\,\text{Log}[n]-b^2\,\text{Log}[n]-n\,\text{Log}[n]-3\,b\,n\,\text{Log}[n]-\frac{3}{2}\,n^2\,\text{Log}[n]-\text{HurwitzZeta}[-1,\,1+b]\,\text{Log}[n]-n\,\text{HurwitzZeta}[0,\,1+b]\,\text{Log}[n]+\frac{1}{2}\,b\,\text{Log}[2\,n]+b^2\,\text{Log}[2\,n]+n\,\text{Log}[2\,n]+4\,b\,n\,\text{Log}[2\,n]+2\,n^2\,\text{Log}[2\,n]+\text{HurwitzZeta}[-1,\,1+b]\,\text{Log}[2\,n]+2\,n\,\text{HurwitzZeta}[0,\,1+b]\,\text{Log}[2\,n]+\text{PoincareSum}\left[\frac{2\,(-1)^m\,n^{-m}\,\text{HurwitzZeta}[-1-m,\,1+b]}{1+m},\,\{m,\,1,\,\infty\}\right]+\text{PoincareSum}\left[\frac{(-1)^{1+m}\,2^{-m}\,n^{-m}\,\text{HurwitzZeta}[-1-m,\,1+b]}{1+m},\,\{m,\,1,\,\infty\}\right]+\text{PoincareSum}\left[\frac{(-1)^m\,n^{-m}\,\text{HurwitzZeta}[-1-m,\,1+b]}{m\,(1+m)},\,\{m,\,1,\,\infty\}\right]+\text{PoincareSum}\left[\frac{(-1)^{1+m}\,2^{-m}\,n^{-m}\,\text{HurwitzZeta}[-1-m,\,1+b]}{m\,(1+m)},\,\{m,\,1,\,\infty\}\right]+\text{PoincareSum}\left[\frac{(-1)^{-1+m}\,b\,n^{-m}\,\text{HurwitzZeta}[-m,\,1+b]}{m},\,\{m,\,1,\,\infty\}\right]+\text{PoincareSum}\left[\frac{\left(-\frac{1}{2}\right)^m\,b\,n^{-m}\,\text{HurwitzZeta}[-m,\,1+b]}{m},\,\{m,\,1,\,\infty\}\right]$

In[ ]:= **TMP04 //. PoincareSumCollect**

Out[ ]= $-b\,n-\frac{5\,n^2}{4}+\text{HurwitzZeta}[-1,\,1+b]-\frac{1}{2}\,b\,\text{Log}[n]-b^2\,\text{Log}[n]-n\,\text{Log}[n]-3\,b\,n\,\text{Log}[n]-\frac{3}{2}\,n^2\,\text{Log}[n]-\text{HurwitzZeta}[-1,\,1+b]\,\text{Log}[n]-n\,\text{HurwitzZeta}[0,\,1+b]\,\text{Log}[n]+\frac{1}{2}\,b\,\text{Log}[2\,n]+b^2\,\text{Log}[2\,n]+n\,\text{Log}[2\,n]+4\,b\,n\,\text{Log}[2\,n]+2\,n^2\,\text{Log}[2\,n]+\text{HurwitzZeta}[-1,\,1+b]\,\text{Log}[2\,n]+2\,n\,\text{HurwitzZeta}[0,\,1+b]\,\text{Log}[2\,n]+\text{PoincareSum}\left[\frac{2\,(-1)^m\,n^{-m}\,\text{HurwitzZeta}[-1-m,\,1+b]}{1+m}+\frac{(-1)^{1+m}\,2^{-m}\,n^{-m}\,\text{HurwitzZeta}[-1-m,\,1+b]}{1+m}+\frac{(-1)^m\,n^{-m}\,\text{HurwitzZeta}[-1-m,\,1+b]}{m\,(1+m)}+\frac{(-1)^{1+m}\,2^{-m}\,n^{-m}\,\text{HurwitzZeta}[-1-m,\,1+b]}{m\,(1+m)}+\frac{(-1)^{-1+m}\,b\,n^{-m}\,\text{HurwitzZeta}[-m,\,1+b]}{m}+\frac{\left(-\frac{1}{2}\right)^m\,b\,n^{-m}\,\text{HurwitzZeta}[-m,\,1+b]}{m},\,\{m,\,1,\,\infty\}\right]$

In[ ]:=
```
TMP = 2 (-1)^m n^-m HurwitzZeta[-1 - m, 1 + b]/(1 + m) + (-1)^(1+m) 2^-m n^-m HurwitzZeta[-1 - m, 1 + b]/(1 + m) +
  (-1)^m n^-m HurwitzZeta[-1 - m, 1 + b]/(m (1 + m)) + (-1)^(1+m) 2^-m n^-m HurwitzZeta[-1 - m, 1 + b]/(m (1 + m)) +
  (-1)^(-1+m) b n^-m HurwitzZeta[-m, 1 + b]/m + (-1/2)^m b n^-m HurwitzZeta[-m, 1 + b]/m //
 FullSimplify[#, Assumptions -> {m ∈ Integers}] &
```

Out[ ]=
$$\frac{1}{m(1+m)}\left(-\frac{1}{2}\right)^m n^{-m}\left(\left(-1+2^m+\left(-1+2^{1+m}\right)m\right)\text{HurwitzZeta}[-1-m,\ 1+b]-\left(-1+2^m\right)b\,(1+m)\,\text{HurwitzZeta}[-m,\ 1+b]\right)$$

In[ ]:=
```
TMP == (-1)^m/m ( ((2-2^-m)/(1-2^-m) m + 1)/(m + 1) HurwitzZeta[-m - 1, b + 1] - b HurwitzZeta[-m, b + 1]) (1 - 2^-m) n^-m //
 FullSimplify
```

Out[ ]= True

In[ ]:=
```
TMP05 = -b n - 5 n^2/4 + HurwitzZeta[-1, 1 + b] - 1/2 b Log[n] - b^2 Log[n] - n Log[n] - 3 b n Log[n] -
  3/2 n^2 Log[n] - HurwitzZeta[-1, 1 + b] Log[n] - n HurwitzZeta[0, 1 + b] Log[n] +
  1/2 b Log[2 n] + b^2 Log[2 n] + n Log[2 n] + 4 b n Log[2 n] + 2 n^2 Log[2 n] +
  HurwitzZeta[-1, 1 + b] Log[2 n] + 2 n HurwitzZeta[0, 1 + b] Log[2 n] + PoincareSum[(-1)^m/m
   ( ((2-2^-m)/(1-2^-m) m + 1)/(m + 1) HurwitzZeta[-m - 1, b + 1] - b HurwitzZeta[-m, b + 1]) (1 - 2^-m) n^-m, {m, 1, ∞}];
```

Leading terms

In[ ]:=
```
TMP = -b n - 5 n^2/4 + HurwitzZeta[-1, 1 + b] - 1/2 b Log[n] - b^2 Log[n] - n Log[n] - 3 b n Log[n] -
  3/2 n^2 Log[n] - HurwitzZeta[-1, 1 + b] Log[n] - n HurwitzZeta[0, 1 + b] Log[n] +
  1/2 b Log[2 n] + b^2 Log[2 n] + n Log[2 n] + 4 b n Log[2 n] + 2 n^2 Log[2 n] +
  HurwitzZeta[-1, 1 + b] Log[2 n] + 2 n HurwitzZeta[0, 1 + b] Log[2 n] // Simplify
```

Out[ ]=
$$-b\,n - \frac{5 n^2}{4} + \frac{1}{2} b\,\mathrm{Log}[2] + b^2\,\mathrm{Log}[2] + n\,\mathrm{Log}[2] + \mathrm{HurwitzZeta}[-1,\ 1+b]\,(1+\mathrm{Log}[2]) + n^2\,\mathrm{Log}[4] + b\,n\,\mathrm{Log}[16] + b\,n\,\mathrm{Log}[n] + \frac{1}{2} n^2\,\mathrm{Log}[n] + n\,\mathrm{HurwitzZeta}[0,\ 1+b]\,\mathrm{Log}[4\,n]$$

Auxiliary results

In[ ]:=
```
-5 n^2/4 + n^2 Log[4] // Factor
```

Out[ ]=
$$\frac{1}{4} n^2\,(-5 + 4\,\mathrm{Log}[4])$$

In[ ]:=
```
+b n Log[n] + n HurwitzZeta[0, 1 + b] Log[n] // FullSimplify
```

Out[ ]=
$$-\frac{1}{2} n\,\mathrm{Log}[n]$$

In[ ]:=
```
-b n + n Log[2] + b n Log[16] + n HurwitzZeta[0, 1 + b] Log[4 ] // FullSimplify
```

Out[ ]=
$$b\,n\,(-1 + \mathrm{Log}[4])$$

```
1/2 b Log[2] + b^2 Log[2] + HurwitzZeta[-1, 1 + b] (1 + Log[2]) // FunctionExpand
```

In[ ]:=
```
1/2 b Log[2] + b^2 Log[2] + (1/2 (5/6 + b - (1 + b)^2) (1 + Log[2]) // Distribute)
```

In[ ]:=
```
5/12 + b/2 - 1/2 (1 + b)^2 + (5 Log[2]/12 + b Log[2] + b^2 Log[2] - 1/2 (1 + b)^2 Log[2] // Factor)
```

In[ ]:=
```
(5/12 + b/2 - 1/2 (1 + b)^2 // FullSimplify) + 1/12 (-1 + 6 b^2) Log[2]
```

Out[ ]=
$$\frac{1}{12}(-1 - 6\,b\,(1+b)) + \frac{1}{12}\left(-1 + 6\,b^2\right)\mathrm{Log}[2]$$

In[ ]:=
$$\text{TMP} == \frac{1}{2} n^2 \text{Log}[n] + \left(2\,\text{Log}[2] - \frac{5}{4}\right) n^2 - \frac{1}{2} n\,\text{Log}[n] + (2\,\text{Log}[2] - 1)\, b\, n + \frac{1}{2}\left(b^2 - \frac{1}{6}\right)\text{Log}[2] - \frac{1}{2}\left(b\,(b+1) + \frac{1}{6}\right) \text{// FullSimplify}$$

Out[ ]= True

$$\text{TMP06} = \frac{1}{2} n^2 \text{Log}[n] + \left(2\,\text{Log}[2] - \frac{5}{4}\right) n^2 - \frac{1}{2} n\,\text{Log}[n] + (2\,\text{Log}[2] - 1)\, b\, n + \frac{1}{2}\left(b^2 - \frac{1}{6}\right)\text{Log}[2] - \frac{1}{2}\left(b\,(b+1) + \frac{1}{6}\right) + \text{PoincareSum}\Big[\frac{(-1)^m}{m}\left(\frac{\frac{2-2^{-m}}{1-2^{-m}}\, m + 1}{m+1}\,\text{HurwitzZeta}[-m-1,\, b+1] - b\,\text{HurwitzZeta}[-m,\, b+1]\right)\left(1 - 2^{-m}\right) n^{-m},\ \{m,\, 1,\, \infty\}\Big];$$

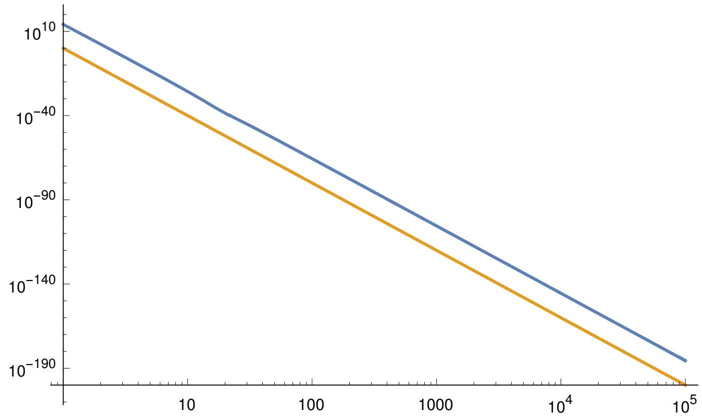
Formula

In[ ]:=
$$\text{ASYMPFracC}[n\_,\, b\_] := \frac{1}{2} n^2 \text{Log}[n] + \left(2\,\text{Log}[2] - \frac{5}{4}\right) n^2 - \frac{1}{2} n\,\text{Log}[n] + (2\,\text{Log}[2] - 1)\, b\, n + \frac{1}{2}\left(b^2 - \frac{1}{6}\right)\text{Log}[2] - \frac{1}{2}\left(b\,(b+1) + \frac{1}{6}\right) + \text{PoincareSum}\Big[\frac{(-1)^m}{m}\left(\frac{\frac{2-2^{-m}}{1-2^{-m}}\, m + 1}{m+1}\,\text{HurwitzZeta}[-m-1,\, b+1] - b\,\text{HurwitzZeta}[-m,\, b+1]\right)\left(1 - 2^{-m}\right) n^{-m},\ \{m,\, 1,\, \infty\}\Big];$$

Cross Check

```
In[ ]:= REF = (2 n + b) Log[Pochhammer[n + b + 1, n]] - Zeta^(1,0)[-1, 2 n + b + 1] + Zeta^(1,0)[-1, n + b + 1];

K = 40;
b = e - 1;

ASYMP = ASYMPFracC[n, b] /. PoincareSumNormalize[K - 1];

LogLogPlot[{Abs[REF - ASYMP], n^-K}, {n, 1, 100 000}, WorkingPrecision → 256]

Clear[b, K];
```

Out[ ]=



Asymptotics : Log $D_n^{(\alpha,\beta)}$

Application of asymptotics

```
- n (n - 1) Log[2] + FracA2[n] + FracB2[n, α] + FracB2[n, β] + FracC2[n, α + β];
(* for comparison *)
```

In[ ]:= **TMP01 = - n (n - 1) Log[2] + ASYMPFracA[n] + ASYMPFracB[n, α] + ASYMPFracB[n, β] + ASYMPFracC[n, α + β]**

Out[ ]=
$$-\frac{1}{6}+\frac{5 n^2}{4}+n(1+\alpha)+n(1+\beta)+\frac{1}{2}\left(-\frac{1}{6}-(\alpha+\beta)(1+\alpha+\beta)\right)-(-1+n)\,n\,\mathrm{Log}[2]+\frac{1}{2}\left(-\frac{1}{6}+(\alpha+\beta)^2\right)\mathrm{Log}[2]+$$
$$n^2\left(-\frac{5}{4}+2\,\mathrm{Log}[2]\right)+n(\alpha+\beta)(-1+2\,\mathrm{Log}[2])+3\,\mathrm{Log}[\mathrm{Glaisher}]+\frac{13\,\mathrm{Log}[n]}{12}+\frac{1}{2}\left(\frac{1}{6}-(1+\alpha)^2\right)\mathrm{Log}[n]+$$
$$\frac{1}{2}\left(\frac{1}{6}-(1+\beta)^2\right)\mathrm{Log}[n]+\mathrm{Log}[2\pi]-n(2+\mathrm{Log}[2\pi])+(1+\alpha)\,\mathrm{Log}[\mathrm{Gamma}[1+\alpha]]+(1+\beta)\,\mathrm{Log}[\mathrm{Gamma}[1+\beta]]+$$
$$\mathrm{PoincareSum}\left[\frac{(-1)^{-1+m}\,n^{-m}\left(-\frac{\mathrm{HurwitzZeta}[-1-m,1+\alpha]}{1+m}+(1+\alpha)\,\mathrm{HurwitzZeta}[-m,\,1+\alpha]\right)}{m},\,\{m,\,1,\,\infty\}\right]+$$
$$\mathrm{PoincareSum}\left[\frac{(-1)^{-1+m}\,n^{-m}\left(-\frac{\mathrm{HurwitzZeta}[-1-m,1+\beta]}{1+m}+(1+\beta)\,\mathrm{HurwitzZeta}[-m,\,1+\beta]\right)}{m},\,\{m,\,1,\,\infty\}\right]+$$
$$\mathrm{PoincareSum}\left[\frac{1}{m}(-1)^m\left(1-2^{-m}\right)n^{-m}\left(\frac{\left(1+\frac{(2-2^{-m})\,m}{1-2^{-m}}\right)\mathrm{HurwitzZeta}[-1-m,\,1+\alpha+\beta]}{1+m}-(\alpha+\beta)\,\mathrm{HurwitzZeta}[-m,\,1+\alpha+\beta]\right),\,\{m,\,1,\,\infty\}\right]+$$
$$\mathrm{PoincareSum}\left[\frac{(-1)^m\,n^{-m}\left(\frac{(1+2\,m)\,\mathrm{Zeta}[-1-m]}{1+m}+2\,\mathrm{Zeta}[-m]\right)}{m},\,\{m,\,1,\,\infty\}\right]-$$
$$\mathrm{PolyGamma}[-2,\,1+\alpha]-\mathrm{PolyGamma}[-2,\,1+\beta]$$

Simplification

In[ ]:= `TMP01 //. PoincareSumCollect`

Out[ ]=
$$-\frac{1}{6}+\frac{5n^2}{4}+n(1+\alpha)+n(1+\beta)+\frac{1}{2}\left(-\frac{1}{6}-(\alpha+\beta)(1+\alpha+\beta)\right)-(-1+n)\,n\,\text{Log}[2]+\frac{1}{2}\left(-\frac{1}{6}+(\alpha+\beta)^2\right)\text{Log}[2]+ n^2\left(-\frac{5}{4}+2\,\text{Log}[2]\right)+n(\alpha+\beta)(-1+2\,\text{Log}[2])+3\,\text{Log}[\text{Glaisher}]+\frac{13\,\text{Log}[n]}{12}+\frac{1}{2}\left(\frac{1}{6}-(1+\alpha)^2\right)\text{Log}[n]+ \frac{1}{2}\left(\frac{1}{6}-(1+\beta)^2\right)\text{Log}[n]+\text{Log}[2\,\pi]-n(2+\text{Log}[2\,\pi])+(1+\alpha)\,\text{Log}[\text{Gamma}[1+\alpha]]+(1+\beta)\,\text{Log}[\text{Gamma}[1+\beta]]+ \text{PoincareSum}\Bigg[\frac{(-1)^{-1+m}\,n^{-m}\left(-\frac{\text{HurwitzZeta}[-1-m,1+\alpha]}{1+m}+(1+\alpha)\,\text{HurwitzZeta}[-m,\,1+\alpha]\right)}{m}+ \frac{(-1)^{-1+m}\,n^{-m}\left(-\frac{\text{HurwitzZeta}[-1-m,1+\beta]}{1+m}+(1+\beta)\,\text{HurwitzZeta}[-m,\,1+\beta]\right)}{m}+\frac{1}{m}(-1)^m\left(1-2^{-m}\right)n^{-m} \left(\frac{\left(1+\frac{(2-2^{-m})\,m}{1-2^{-m}}\right)\text{HurwitzZeta}[-1-m,\,1+\alpha+\beta]}{1+m}-(\alpha+\beta)\,\text{HurwitzZeta}[-m,\,1+\alpha+\beta]\right)+ \frac{(-1)^m\,n^{-m}\left(\frac{(1+2\,m)\,\text{Zeta}[-1-m]}{1+m}+2\,\text{Zeta}[-m]\right)}{m},\ \{m,\,1,\,\infty\}\Bigg]-\text{PolyGamma}[-2,\,1+\alpha]-\text{PolyGamma}[-2,\,1+\beta]$$

In[ ]:= TMP02 = $-\frac{1}{6}+\frac{5n^2}{4}+n(1+\alpha)+n(1+\beta)+\frac{1}{2}\left(-\frac{1}{6}-(\alpha+\beta)(1+\alpha+\beta)\right)-(-1+n)\,n\,\text{Log}[2]+$
$\frac{1}{2}\left(-\frac{1}{6}+(\alpha+\beta)^2\right)\text{Log}[2]+n^2\left(-\frac{5}{4}+2\,\text{Log}[2]\right)+n(\alpha+\beta)(-1+2\,\text{Log}[2])+3\,\text{Log}[\text{Glaisher}]+$
$\frac{13\,\text{Log}[n]}{12}+\frac{1}{2}\left(\frac{1}{6}-(1+\alpha)^2\right)\text{Log}[n]+\frac{1}{2}\left(\frac{1}{6}-(1+\beta)^2\right)\text{Log}[n]+\text{Log}[2\pi]-n(2+\text{Log}[2\pi])+$
$(1+\alpha)\,\text{Log}[\text{Gamma}[1+\alpha]]+(1+\beta)\,\text{Log}[\text{Gamma}[1+\beta]]-\text{PolyGamma}[-2,1+\alpha]-\text{PolyGamma}[-2,1+\beta]+$
$\text{PoincareSum}\Big[\frac{(-1)^{-1+m}n^{-m}\left(-\frac{\text{HurwitzZeta}[-1-m,1+\alpha]}{1+m}+(1+\alpha)\,\text{HurwitzZeta}[-m,1+\alpha]\right)}{m}+$
$\frac{(-1)^{-1+m}n^{-m}\left(-\frac{\text{HurwitzZeta}[-1-m,1+\beta]}{1+m}+(1+\beta)\,\text{HurwitzZeta}[-m,1+\beta]\right)}{m}+\frac{1}{m}(-1)^m\left(1-2^{-m}\right)$
$n^{-m}\left(\frac{\left(1+\frac{(2-2^{-m})\,m}{1-2^{-m}}\right)\text{HurwitzZeta}[-1-m,1+\alpha+\beta]}{1+m}-(\alpha+\beta)\,\text{HurwitzZeta}[-m,1+\alpha+\beta]\right)+$
$\frac{(-1)^m n^{-m}\left(\frac{(1+2m)\,\text{Zeta}[-1-m]}{1+m}+2\,\text{Zeta}[-m]\right)}{m},\{m,1,\infty\}\Big];$

In[ ]:= **TMP =**

$$\frac{(-1)^{-1+m} n^{-m} \left(-\frac{\text{HurwitzZeta}[-1-m,1+\alpha]}{1+m} + (1+\alpha)\,\text{HurwitzZeta}[-m, 1+\alpha]\right)}{m} + \frac{(-1)^{-1+m} n^{-m} \left(-\frac{\text{HurwitzZeta}[-1-m,1+\beta]}{1+m} + (1+\beta)\,\text{HurwitzZeta}[-m, 1+\beta]\right)}{m} + \frac{1}{m}(-1)^m\left(1-2^{-m}\right) n^{-m}\left(\frac{\left(1+\frac{(2-2^{-m})\,m}{1-2^{-m}}\right)\text{HurwitzZeta}[-1-m, 1+\alpha+\beta]}{1+m} - (\alpha+\beta)\,\text{HurwitzZeta}[-m, 1+\alpha+\beta]\right) + \frac{(-1)^m n^{-m}\left(\frac{(1+2\,m)\,\text{Zeta}[-1-m]}{1+m} + 2\,\text{Zeta}[-m]\right)}{m}$$

// Simplify

Out[ ]=

$$\frac{1}{m\,(1+m)}(-1)^m n^{-m}\Big(\text{HurwitzZeta}[-1-m, 1+\alpha] + \text{HurwitzZeta}[-1-m, 1+\beta] + 2^{-m}\left(-1+2^m+\left(-1+2^{1+m}\right)m\right)\text{HurwitzZeta}[-1-m, 1+\alpha+\beta] - (1+m)(1+\alpha)\,\text{HurwitzZeta}[-m, 1+\alpha] - (1+m)(1+\beta)\,\text{HurwitzZeta}[-m, 1+\beta] - \left(1-2^{-m}\right)(1+m)(\alpha+\beta)\,\text{HurwitzZeta}[-m, 1+\alpha+\beta] + (1+2\,m)\,\text{Zeta}[-1-m] + 2\,(1+m)\,\text{Zeta}[-m]\Big)$$

In[ ]:=

$$-\frac{1}{(1+m)}\Big(\text{HurwitzZeta}[-1-m, 1+\alpha] + \text{HurwitzZeta}[-1-m, 1+\beta] + 2^{-m}\left(-1+2^m+\left(-1+2^{1+m}\right)m\right)\text{HurwitzZeta}[-1-m, 1+\alpha+\beta] - (1+m)(1+\alpha)\,\text{HurwitzZeta}[-m, 1+\alpha] - (1+m)(1+\beta)\,\text{HurwitzZeta}[-m, 1+\beta] - \left(1-2^{-m}\right)(1+m)(\alpha+\beta)\,\text{HurwitzZeta}[-m, 1+\alpha+\beta] + (1+2\,m)\,\text{Zeta}[-1-m] + 2\,(1+m)\,\text{Zeta}[-m]\Big)$$

// Distribute

Out[ ]=

$$-\frac{\text{HurwitzZeta}[-1-m, 1+\alpha]}{1+m} - \frac{\text{HurwitzZeta}[-1-m, 1+\beta]}{1+m} - \frac{2^{-m}\left(-1+2^m+\left(-1+2^{1+m}\right)m\right)\text{HurwitzZeta}[-1-m, 1+\alpha+\beta]}{1+m} + (1+\alpha)\,\text{HurwitzZeta}[-m, 1+\alpha] + (1+\beta)\,\text{HurwitzZeta}[-m, 1+\beta] + \left(1-2^{-m}\right)(\alpha+\beta)\,\text{HurwitzZeta}[-m, 1+\alpha+\beta] - \frac{(1+2\,m)\,\text{Zeta}[-1-m]}{1+m} - 2\,\text{Zeta}[-m]$$

In[ ]:= $2^{-m}\left(-1+2^m+\left(-1+2^{1+m}\right)m\right)$ **// Distribute**

In[ ]:= $1-2^{-m}+\left(2^{-m}\left(-1+2^{1+m}\right)\ \text{// Distribute}\right)m$

Out[ ]= $1-2^{-m}+\left(2-2^{-m}\right)m$

In[ ]:= $\text{TMP} == \frac{(-1)^{m-1}}{m}\left(-\frac{2m+1}{m+1}\,\text{Zeta}[-m-1] - 2\,\text{Zeta}[-m] + (\alpha+1)\,\text{HurwitzZeta}[-m,\alpha+1] - \frac{\text{HurwitzZeta}[-m-1,\alpha+1]}{m+1} + (\beta+1)\,\text{HurwitzZeta}[-m,\beta+1] - \frac{\text{HurwitzZeta}[-m-1,\beta+1]}{m+1} - \frac{(2-2^{-m})\,m+1-2^{-m}}{m+1}\,\text{HurwitzZeta}[-m-1,\alpha+\beta+1] + (\alpha+\beta)\,(1-2^{-m})\,\text{HurwitzZeta}[-m,\alpha+\beta+1]\right) n^{-m}$ // FullSimplify

Out[ ]= True

In[ ]:= $\text{TMP03} = -\frac{1}{6} + \frac{5n^2}{4} + n\,(1+\alpha) + n\,(1+\beta) + \frac{1}{2}\left(-\frac{1}{6} - (\alpha+\beta)\,(1+\alpha+\beta)\right) - (-1+n)\,n\,\text{Log}[2] + \frac{1}{2}\left(-\frac{1}{6} + (\alpha+\beta)^2\right)\text{Log}[2] + n^2\left(-\frac{5}{4} + 2\,\text{Log}[2]\right) + n\,(\alpha+\beta)\,(-1+2\,\text{Log}[2]) + 3\,\text{Log}[\text{Glaisher}] + \frac{13\,\text{Log}[n]}{12} + \frac{1}{2}\left(\frac{1}{6} - (1+\alpha)^2\right)\text{Log}[n] + \frac{1}{2}\left(\frac{1}{6} - (1+\beta)^2\right)\text{Log}[n] + \text{Log}[2\pi] - n\,(2+\text{Log}[2\pi]) + (1+\alpha)\,\text{Log}[\text{Gamma}[1+\alpha]] + (1+\beta)\,\text{Log}[\text{Gamma}[1+\beta]] - \text{PolyGamma}[-2,1+\alpha] - \text{PolyGamma}[-2,1+\beta] + \text{PoincareSum}\Big[\frac{(-1)^{m-1}}{m}\left(-\frac{2m+1}{m+1}\,\text{Zeta}[-m-1] - 2\,\text{Zeta}[-m] + (\alpha+1)\,\text{HurwitzZeta}[-m,\alpha+1] - \frac{\text{HurwitzZeta}[-m-1,\alpha+1]}{m+1} + (\beta+1)\,\text{HurwitzZeta}[-m,\beta+1] - \frac{\text{HurwitzZeta}[-m-1,\beta+1]}{m+1} - \frac{(2-2^{-m})\,m+1-2^{-m}}{m+1}\,\text{HurwitzZeta}[-m-1,\alpha+\beta+1] + (\alpha+\beta)\,(1-2^{-m})\,\text{HurwitzZeta}[-m,\alpha+\beta+1]\right) n^{-m}, \{m, 1, \infty\}\Big];$

Leading terms

In[ ]:=
```
TMP = -1/6 + 5 n^2/4 + n (1 + α) + n (1 + β) + 1/2 (-1/6 - (α + β) (1 + α + β)) - (-1 + n) n Log[2] +
  1/2 (-1/6 + (α + β)^2) Log[2] + n^2 (-5/4 + 2 Log[2]) + n (α + β) (-1 + 2 Log[2]) + 3 Log[Glaisher] +
  13 Log[n]/12 + 1/2 (1/6 - (1 + α)^2) Log[n] + 1/2 (1/6 - (1 + β)^2) Log[n] + Log[2 π] - n (2 + Log[2 π]) +
  (1 + α) Log[Gamma[1 + α]] + (1 + β) Log[Gamma[1 + β]] - PolyGamma[-2, 1 + α] - PolyGamma[-2, 1 + β]
```

Out[ ]=

$$-\frac{1}{6}+\frac{5 n^2}{4}+n(1+\alpha)+n(1+\beta)+\frac{1}{2}\left(-\frac{1}{6}-(\alpha+\beta)(1+\alpha+\beta)\right)-(-1+n)\,n\,\mathrm{Log}[2]+\frac{1}{2}\left(-\frac{1}{6}+(\alpha+\beta)^2\right)\mathrm{Log}[2]+n^2\left(-\frac{5}{4}+2\,\mathrm{Log}[2]\right)+n(\alpha+\beta)(-1+2\,\mathrm{Log}[2])+3\,\mathrm{Log}[\mathrm{Glaisher}]+\frac{13\,\mathrm{Log}[n]}{12}+\frac{1}{2}\left(\frac{1}{6}-(1+\alpha)^2\right)\mathrm{Log}[n]+\frac{1}{2}\left(\frac{1}{6}-(1+\beta)^2\right)\mathrm{Log}[n]+\mathrm{Log}[2\pi]-n(2+\mathrm{Log}[2\pi])+(1+\alpha)\,\mathrm{Log}[\mathrm{Gamma}[1+\alpha]]+(1+\beta)\,\mathrm{Log}[\mathrm{Gamma}[1+\beta]]-\mathrm{PolyGamma}[-2,1+\alpha]-\mathrm{PolyGamma}[-2,1+\beta]$$

Auxiliary results

In[ ]:=
```
+ 5 n^2/4 - (n) n Log[2] + n^2 (-5/4 + 2 Log[2]) // Simplify
```

Out[ ]= $n^2\,\mathrm{Log}[2]$

In[ ]:=
```
+n (1 + α) + n (1 + β) - (-1) n Log[2] + n (α + β) (-1 + 2 Log[2]) - n (2 + Log[2 π]) // FullSimplify // Factor
```

In[ ]:=
```
n (α Log[4] + β Log[4] - Log[π] // FullSimplify)
```

Out[ ]= $n\,((\alpha+\beta)\,\mathrm{Log}[4]-\mathrm{Log}[\pi])$

In[ ]:=
```
+ 13 Log[n]/12 + 1/2 (1/6 - (1 + α)^2) Log[n] + 1/2 (1/6 - (1 + β)^2) Log[n];
```

In[ ]:=
```
(13/12 + 1/2 (1/6 - (1 + α)^2) + 1/2 (1/6 - (1 + β)^2)) Log[n];
```

In[ ]:=
```
(13/12 + 1/2 × 1/6 - 1/2 (1 + α)^2 + 1/2 × 1/6 - 1/2 (1 + β)^2) Log[n]
```

Out[ ]= $\left(\frac{5}{4}-\frac{1}{2}(1+\alpha)^2-\frac{1}{2}(1+\beta)^2\right)\mathrm{Log}[n]$

In[ ]:=
```
-1/6 + 1/2 (-1/6 - (α + β) (1 + α + β)) == -1/8 - 1/2 (α + β + 1/2)^2 // FullSimplify
```

Out[ ]= True

In[ ]:=
```
TMP == Log[2] n^2 + (2 (α + β) Log[2] - Log[π]) n + 1/2 (5/2 - (α + 1)^2 - (β + 1)^2) Log[n] - 1/8 - 1/2 (α + β + 1/2)^2 +
  1/2 (11/6 + (α + β)^2) Log[2] + Log[ π] + 3 Log[Glaisher] + (α + 1) Log[Gamma[α + 1]] -
  PolyGamma[-2, α + 1] + (β + 1) Log[Gamma[β + 1]] - PolyGamma[-2, β + 1] // FullSimplify
```

Out[ ]= True

```
TMP04 = Log[2] n^2 + (2 (α + β) Log[2] - Log[π]) n + 1/2 (5/2 - (α + 1)^2 - (β + 1)^2) Log[n] - 1/8 -
  1/2 (α + β + 1/2)^2 + 1/2 (11/6 + (α + β)^2) Log[2] + Log[ π] + 3 Log[Glaisher] + (α + 1) Log[Gamma[α + 1]] -
  PolyGamma[-2, α + 1] + (β + 1) Log[Gamma[β + 1]] - PolyGamma[-2, β + 1] +
  PoincareSum[(-1)^(m-1)/m (-(2 m + 1)/(m + 1) Zeta[-m - 1] - 2 Zeta[-m] + (α + 1) HurwitzZeta[-m, α + 1] -
      HurwitzZeta[-m - 1, α + 1]/(m + 1) + (β + 1) HurwitzZeta[-m, β + 1] -
      HurwitzZeta[-m - 1, β + 1]/(m + 1) - ((2 - 2^-m) m + 1 - 2^-m)/(m + 1) HurwitzZeta[-m - 1, α + β + 1] +
     (α + β) (1 - 2^-m) HurwitzZeta[-m, α + β + 1]) n^-m, {m, 1, ∞}];
```

Formula

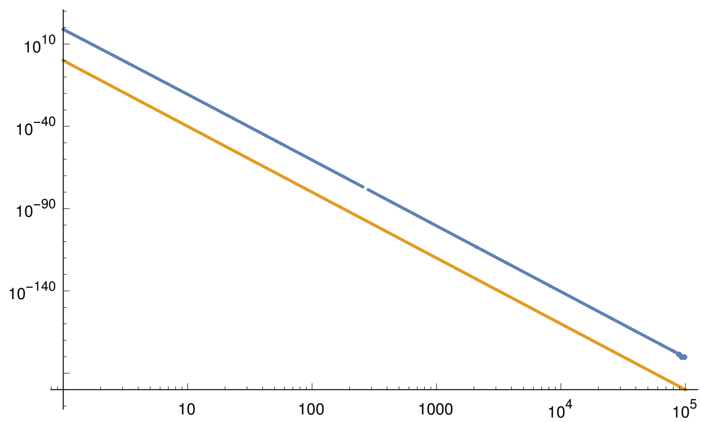
In[ ]:=

$$\text{ASYMPD}[n\_, \alpha\_, \beta\_] := \text{Log}[2]\, n^2 + (2(\alpha+\beta)\text{Log}[2] - \text{Log}[\pi])\, n + \frac{1}{2}\left(\frac{5}{2} - (\alpha+1)^2 - (\beta+1)^2\right)\text{Log}[n] - \frac{1}{8} - \frac{1}{2}\left(\alpha+\beta+\frac{1}{2}\right)^2 + \frac{1}{2}\left(\frac{11}{6} + (\alpha+\beta)^2\right)\text{Log}[2] + \text{Log}[\pi] + 3\,\text{Log}[\text{Glaisher}] + (\alpha+1)\,\text{Log}[\text{Gamma}[\alpha+1]] - \text{PolyGamma}[-2, \alpha+1] + (\beta+1)\,\text{Log}[\text{Gamma}[\beta+1]] - \text{PolyGamma}[-2, \beta+1] + \text{PoincareSum}\left[\frac{(-1)^{m-1}}{m}\left(-\frac{2m+1}{m+1}\text{Zeta}[-m-1] - 2\,\text{Zeta}[-m] + (\alpha+1)\,\text{HurwitzZeta}[-m, \alpha+1] - \frac{\text{HurwitzZeta}[-m-1, \alpha+1]}{m+1} + (\beta+1)\,\text{HurwitzZeta}[-m, \beta+1] - \frac{\text{HurwitzZeta}[-m-1, \beta+1]}{m+1} - \frac{(2-2^{-m})\,m + 1 - 2^{-m}}{m+1}\text{HurwitzZeta}[-m-1, \alpha+\beta+1] + (\alpha+\beta)\left(1-2^{-m}\right)\text{HurwitzZeta}[-m, \alpha+\beta+1]\right) n^{-m}, \{m, 1, \infty\}\right];$$

Cross Check

In[ ]:=
```
REF = - n (n - 1) Log[2] + FracA2[n] + FracB2[n, α] + FracB2[n, β] + FracC2[n, α + β];

K = 40;
α = 1/Sqrt[π];
β = ⅇ;

ASYMP = ASYMPD[n, α, β] /. PoincareSumNormalize[K - 1];

LogLogPlot[{Abs[REF - ASYMP], n^-K}, {n, 1, 100 000}, WorkingPrecision → 512]

Clear[α, β, K];
```

Out[ ]=



Section 3

Proof of Theorem 1.1

Verification of starting point

Symbolic Cross Check (small degree n)

In[ ]:=
```
λ[n_, α_, β_] := 2^-n Binomial[2 n + α + β, n];
Discr[n_, α_, β_] := 2^(-n (n-1)) Product[v^(v-2 n+2) (v + α)^(v-1) (v + β)^(v-1) (v + n + α + β)^(n-v), {v, 1, n}];

n = 2; (* degree 3 already too large for reasonable time of evaluation *)

p = 1 / 2;
q = π;

α = 2 p - 1;
β = 2 q - 1;


zeros = Sort[x /. Solve[JacobiP[n, α, β, x] == 0, x]];

(* cf. (1.4) *)
REF = Product[(1 - zeros[[i]])^p, {i, 1, n}] × Product[Abs[zeros[[j]] - zeros[[k]]],
    {j, 1, n - 1}, {k, j + 1, n}] × Product[(1 + zeros[[ell]])^q, {ell, 1, n}];

RES = (JacobiP[n, α, β, 1]^p / λ[n, α, β]^p) (Sqrt[Discr[n, α, β]] / λ[n, α, β]^(n-1)) (((-1)^n JacobiP[n, α, β, -1])^q / λ[n, α, β]^q);

RES == REF // FullSimplify

Clear[α, β, n, p, q, zeros];
```

Out[ ]= True

Numerical Cross Check (general degree n)

In[ ]:=
```
λ[n_, α_, β_] := 2^-n Binomial[2 n + α + β, n];
Discr[n_, α_, β_] := 2^(-n (n-1)) Product[v^(v-2 n+2) (v + α)^(v-1) (v + β)^(v-1) (v + n + α + β)^(n-v), {v, 1, n}];

Acc = 64;

n = 8;

p = N[1 / 2, Acc];
q = N[π, Acc];

α = 2 p - 1;
β = 2 q - 1;


zeros = Sort[x /. NSolve[JacobiP[n, α, β, x] == 0, x, WorkingPrecision → Acc]];

(* cf. (1.4) *)
REF = Product[(1 - zeros[[i]])^p, {i, 1, n}] × Product[Abs[zeros[[j]] - zeros[[k]]],
    {j, 1, n - 1}, {k, j + 1, n}] × Product[(1 + zeros[[ell]])^q, {ell, 1, n}];

RES = JacobiP[n, α, β, 1]^p / λ[n, α, β]^p  Sqrt[Discr[n, α, β]] / λ[n, α, β]^(n-1)  ((-1)^n JacobiP[n, α, β, -1])^q / λ[n, α, β]^q;

{RES - REF, (RES - REF) / REF}

Clear[α, β, n, p, q, zeros];
```

Out[ ]= $\{0. \times 10^{-69}, 0. \times 10^{-62}\}$

Verification of 2 nd Rmk after Thm 1.1

In[ ]:=
```
λ[n_, α_, β_] := 2^-n Binomial[2 n + α + β, n];
Discr[n_, α_, β_] := 2^(-n (n-1)) Product[ν^(ν-2 n+2) (ν + α)^(ν-1) (ν + β)^(ν-1) (ν + n + α + β)^(n-ν), {ν, 1, n}];

n = 5; (*   *)

p =.; 1 / 2;
q =.; π;

α = 2 p - 1;
β = 2 q - 1;


zeros = Sort[x /. Solve[JacobiP[n, α, β, x] == 0, x]];
```

$$\text{REF} = \left(\frac{\text{JacobiP}[n, \alpha, \beta, 1]^p}{\lambda[n, \alpha, \beta]^p} \frac{\sqrt{\text{Discr}[n, \alpha, \beta]}}{\lambda[n, \alpha, \beta]^{n-1}} \frac{((-1)^n \text{JacobiP}[n, \alpha, \beta, -1])^q}{\lambda[n, \alpha, \beta]^q}\right)^2;$$

$$\text{RES} = 2^{n (n+2 p+2 q-1)} \frac{\text{Product}[k^k (k+2p-1)^{k+2p-1} (k+2q-1)^{k+2q-1}, \{k, 1, n\}]}{\text{Product}[(k+2p+2q)^{k+2p+2q}, \{k, n-1, 2(n-1)\}]};$$

```
RES == REF // FullSimplify[#, Assumptions → {p > 0, q > 0}] &

Clear[α, β, n, p, q, zeros];
```

Out[ ]= True

Application of asymptotics

```
2 (n + p + q - 1) Logλ[n, α, β] - LogD[n, α, β] - 2 p LogP[n, α, β] - 2 q LogP[n, β, α];

ASYMPλ[n, α, β];
ASYMPJacobiPofOne[n, α, β];
ASYMPD[n, α, β];

(* for comparison ... *)
```

```
In[ ]:= α = 2 p - 1;
        β = 2 q - 1;

        TMP01 = 2 (n + p + q - 1) ASYMPλ[n, α, β] - ASYMPD[n, α, β] -
            2 p ASYMPJacobiPofOne[n, α, β] - 2 q ASYMPJacobiPofOne[n, β, α] // Expand

        Clear[α, β];
```

$$\frac{5}{4} - 3p + 2p^2 - 3q + 4pq + 2q^2 + \frac{13\,\mathrm{Log}[2]}{12} - 2n\,\mathrm{Log}[2] + n^2\,\mathrm{Log}[2] - 4p\,\mathrm{Log}[2] + 2np\,\mathrm{Log}[2] + 2p^2\,\mathrm{Log}[2] - 4q\,\mathrm{Log}[2] + 2nq\,\mathrm{Log}[2] + 4pq\,\mathrm{Log}[2] + 2q^2\,\mathrm{Log}[2] - 3\,\mathrm{Log}[\mathrm{Glaisher}] - \frac{\mathrm{Log}[n]}{4} - n\,\mathrm{Log}[n] + p\,\mathrm{Log}[n] - 2p^2\,\mathrm{Log}[n] + q\,\mathrm{Log}[n] - 2q^2\,\mathrm{Log}[n] - p\,\mathrm{Log}[\pi] - q\,\mathrm{Log}[\pi] -$$

$$\mathrm{PoincareSum}\Big[\frac{1}{m}(-1)^{-1+m}n^{-m}\Big(-\frac{\mathrm{HurwitzZeta}[-1-m, 2p]}{1+m} - \frac{\mathrm{HurwitzZeta}[-1-m, 2q]}{1+m} - \frac{(1-2^{-m}+(2-2^{-m})m)\,\mathrm{HurwitzZeta}[-1-m, -1+2p+2q]}{1+m} + 2p\,\mathrm{HurwitzZeta}[-m, 2p] + 2q\,\mathrm{HurwitzZeta}[-m, 2q] + (1-2^{-m})(-2+2p+2q)\,\mathrm{HurwitzZeta}[-m, -1+2p+2q] - \frac{(1+2m)\,\mathrm{Zeta}[-1-m]}{1+m} - 2\,\mathrm{Zeta}[-m]\Big), \{m, 1, \infty\}\Big] -$$

$$2p\,\mathrm{PoincareSum}\Big[\frac{(-1)^m n^{-m}(\mathrm{HurwitzZeta}[-m, 2p] - \mathrm{Zeta}[-m])}{m}, \{m, 1, \infty\}\Big] -$$

$$2q\,\mathrm{PoincareSum}\Big[\frac{(-1)^m n^{-m}(\mathrm{HurwitzZeta}[-m, 2q] - \mathrm{Zeta}[-m])}{m}, \{m, 1, \infty\}\Big] -$$

$$2\,\mathrm{PoincareSum}\Big[\frac{(-1)^{-1+m} n^{-m}((1-2^{-m})\,\mathrm{HurwitzZeta}[-m, -1+2p+2q] + \mathrm{Zeta}[-m])}{m}, \{m, 1, \infty\}\Big] +$$

$$2n\,\mathrm{PoincareSum}\Big[\frac{(-1)^{-1+m} n^{-m}((1-2^{-m})\,\mathrm{HurwitzZeta}[-m, -1+2p+2q] + \mathrm{Zeta}[-m])}{m}, \{m, 1, \infty\}\Big] +$$

$$2p\,\mathrm{PoincareSum}\Big[\frac{(-1)^{-1+m} n^{-m}((1-2^{-m})\,\mathrm{HurwitzZeta}[-m, -1+2p+2q] + \mathrm{Zeta}[-m])}{m}, \{m, 1, \infty\}\Big] +$$

$$2q\,\mathrm{PoincareSum}\Big[\frac{(-1)^{-1+m} n^{-m}((1-2^{-m})\,\mathrm{HurwitzZeta}[-m, -1+2p+2q] + \mathrm{Zeta}[-m])}{m}, \{m, 1, \infty\}\Big] +$$

$$\mathrm{PolyGamma}[-2, 2p] + \mathrm{PolyGamma}[-2, 2q]$$

Simplification

In[ ]:= TMP02 = TMP01 /. PoincareSumFactorUnderSum

Out[ ]=
$$\frac{5}{4} - 3p + 2p^2 - 3q + 4pq + 2q^2 + \frac{13\,\text{Log}[2]}{12} - 2n\,\text{Log}[2] + n^2\,\text{Log}[2] - 4p\,\text{Log}[2] + 2np\,\text{Log}[2] + 2p^2\,\text{Log}[2] - 4q\,\text{Log}[2] + 2nq\,\text{Log}[2] + 4pq\,\text{Log}[2] + 2q^2\,\text{Log}[2] - 3\,\text{Log}[\text{Glaisher}] - \frac{\text{Log}[n]}{4} - n\,\text{Log}[n] + p\,\text{Log}[n] - 2p^2\,\text{Log}[n] + q\,\text{Log}[n] - 2q^2\,\text{Log}[n] - p\,\text{Log}[\pi] - q\,\text{Log}[\pi] +$$
$$\text{PoincareSum}\Big[\frac{1}{m}(-1)^m n^{-m}\Big(-\frac{\text{HurwitzZeta}[-1-m,2p]}{1+m} - \frac{\text{HurwitzZeta}[-1-m,2q]}{1+m} - \frac{\left(1-2^{-m}+\left(2-2^{-m}\right)m\right)\text{HurwitzZeta}[-1-m,-1+2p+2q]}{1+m} + 2p\,\text{HurwitzZeta}[-m,2p] + 2q\,\text{HurwitzZeta}[-m,2q] + \left(1-2^{-m}\right)(-2+2p+2q)\,\text{HurwitzZeta}[-m,-1+2p+2q] - \frac{(1+2m)\,\text{Zeta}[-1-m]}{1+m} - 2\,\text{Zeta}[-m]\Big),\{m,1,\infty\}\Big] +$$
$$\text{PoincareSum}\Big[-\frac{2(-1)^m n^{-m}p\,(\text{HurwitzZeta}[-m,2p]-\text{Zeta}[-m])}{m},\{m,1,\infty\}\Big] +$$
$$\text{PoincareSum}\Big[-\frac{2(-1)^m n^{-m}q\,(\text{HurwitzZeta}[-m,2q]-\text{Zeta}[-m])}{m},\{m,1,\infty\}\Big] +$$
$$\text{PoincareSum}\Big[\frac{2(-1)^{-1+m}n^{1-m}\left(\left(1-2^{-m}\right)\text{HurwitzZeta}[-m,-1+2p+2q]+\text{Zeta}[-m]\right)}{m},\{m,1,\infty\}\Big] +$$
$$\text{PoincareSum}\Big[-\frac{2(-1)^{-1+m}n^{-m}\left(\left(1-2^{-m}\right)\text{HurwitzZeta}[-m,-1+2p+2q]+\text{Zeta}[-m]\right)}{m},\{m,1,\infty\}\Big] +$$
$$\text{PoincareSum}\Big[\frac{2(-1)^{-1+m}n^{-m}p\left(\left(1-2^{-m}\right)\text{HurwitzZeta}[-m,-1+2p+2q]+\text{Zeta}[-m]\right)}{m},\{m,1,\infty\}\Big] +$$
$$\text{PoincareSum}\Big[\frac{2(-1)^{-1+m}n^{-m}q\left(\left(1-2^{-m}\right)\text{HurwitzZeta}[-m,-1+2p+2q]+\text{Zeta}[-m]\right)}{m},\{m,1,\infty\}\Big] +$$
$$\text{PolyGamma}[-2,2p] + \text{PolyGamma}[-2,2q]$$

In[ ]:=

```
TMP03 = 5/4 - 3 p + 2 p^2 - 3 q + 4 p q + 2 q^2 + (13 Log[2])/12 - 2 n Log[2] + n^2 Log[2] -
  4 p Log[2] + 2 n p Log[2] + 2 p^2 Log[2] - 4 q Log[2] + 2 n q Log[2] + 4 p q Log[2] +
  2 q^2 Log[2] - 3 Log[Glaisher] - Log[n]/4 - n Log[n] + p Log[n] - 2 p^2 Log[n] + q Log[n] -
  2 q^2 Log[n] - p Log[π] - q Log[π] + PolyGamma[-2, 2 p] + PolyGamma[-2, 2 q] +
  PoincareSum[1/m (-1)^m n^-m (-HurwitzZeta[-1 - m, 2 p]/(1 + m) - HurwitzZeta[-1 - m, 2 q]/(1 + m) -
      ((1 - 2^-m + (2 - 2^-m) m) HurwitzZeta[-1 - m, -1 + 2 p + 2 q])/(1 + m) + 2 p HurwitzZeta[-m, 2 p] +
      2 q HurwitzZeta[-m, 2 q] + (1 - 2^-m) (-2 + 2 p + 2 q) HurwitzZeta[-m, -1 + 2 p + 2 q] -
      ((1 + 2 m) Zeta[-1 - m])/(1 + m) - 2 Zeta[-m]), {m, 1, ∞}] +
  PoincareSum[-(2 (-1)^m n^-m p (HurwitzZeta[-m, 2 p] - Zeta[-m]))/m, {m, 1, ∞}] +
  PoincareSum[-(2 (-1)^m n^-m q (HurwitzZeta[-m, 2 q] - Zeta[-m]))/m, {m, 1, ∞}] +
  (PoincareSum[(2 (-1)^(-1 + m) n^(1 - m) ((1 - 2^-m) HurwitzZeta[-m, -1 + 2 p + 2 q] + Zeta[-m]))/m, {m, 1, ∞}] /.
    PoincareSumIndexShiftUp[1]) +
  PoincareSum[-(2 (-1)^(-1 + m) n^-m ((1 - 2^-m) HurwitzZeta[-m, -1 + 2 p + 2 q] + Zeta[-m]))/m, {m, 1, ∞}] +
  PoincareSum[(2 (-1)^(-1 + m) n^-m p ((1 - 2^-m) HurwitzZeta[-m, -1 + 2 p + 2 q] + Zeta[-m]))/m, {m, 1, ∞}] +
  PoincareSum[(2 (-1)^(-1 + m) n^-m q ((1 - 2^-m) HurwitzZeta[-m, -1 + 2 p + 2 q] + Zeta[-m]))/m, {m, 1, ∞}]
```

Out[ ]=
$$\frac{5}{4} - 3\,p + 2\,p^2 - 3\,q + 4\,p\,q + 2\,q^2 + \frac{13\,\text{Log}[2]}{12} - 2\,n\,\text{Log}[2] + n^2\,\text{Log}[2] - 4\,p\,\text{Log}[2] + 2\,n\,p\,\text{Log}[2] +$$
$$2\,p^2\,\text{Log}[2] - 4\,q\,\text{Log}[2] + 2\,n\,q\,\text{Log}[2] + 4\,p\,q\,\text{Log}[2] + 2\,q^2\,\text{Log}[2] - 3\,\text{Log}[\text{Glaisher}] -$$
$$\frac{\text{Log}[n]}{4} - n\,\text{Log}[n] + p\,\text{Log}[n] - 2\,p^2\,\text{Log}[n] + q\,\text{Log}[n] - 2\,q^2\,\text{Log}[n] - p\,\text{Log}[\pi] - q\,\text{Log}[\pi] +$$
$$\text{PoincareSum}\left[\frac{2\,(-1)^m\,n^{-m}\left(\left(1 - 2^{-1-m}\right)\text{HurwitzZeta}[-1-m, -1+2\,p+2\,q] + \text{Zeta}[-1-m]\right)}{1+m}, \{m, 0, \infty\}\right] +$$
$$\text{PoincareSum}\left[\frac{1}{m}\,(-1)^m\,n^{-m}\left(-\frac{\text{HurwitzZeta}[-1-m, 2\,p]}{1+m} - \frac{\text{HurwitzZeta}[-1-m, 2\,q]}{1+m} - \right.\right.$$
$$\frac{\left(1 - 2^{-m} + \left(2 - 2^{-m}\right)m\right)\text{HurwitzZeta}[-1-m, -1+2\,p+2\,q]}{1+m} + 2\,p\,\text{HurwitzZeta}[-m, 2\,p] +$$
$$2\,q\,\text{HurwitzZeta}[-m, 2\,q] + \left(1 - 2^{-m}\right)(-2+2\,p+2\,q)\,\text{HurwitzZeta}[-m, -1+2\,p+2\,q] -$$
$$\left.\left.\frac{(1+2\,m)\,\text{Zeta}[-1-m]}{1+m} - 2\,\text{Zeta}[-m]\right), \{m, 1, \infty\}\right] +$$
$$\text{PoincareSum}\left[-\frac{2\,(-1)^m\,n^{-m}\,p\,(\text{HurwitzZeta}[-m, 2\,p] - \text{Zeta}[-m])}{m}, \{m, 1, \infty\}\right] +$$
$$\text{PoincareSum}\left[-\frac{2\,(-1)^m\,n^{-m}\,q\,(\text{HurwitzZeta}[-m, 2\,q] - \text{Zeta}[-m])}{m}, \{m, 1, \infty\}\right] +$$
$$\text{PoincareSum}\left[-\frac{2\,(-1)^{-1+m}\,n^{-m}\left(\left(1 - 2^{-m}\right)\text{HurwitzZeta}[-m, -1+2\,p+2\,q] + \text{Zeta}[-m]\right)}{m}, \{m, 1, \infty\}\right] +$$
$$\text{PoincareSum}\left[\frac{2\,(-1)^{-1+m}\,n^{-m}\,p\left(\left(1 - 2^{-m}\right)\text{HurwitzZeta}[-m, -1+2\,p+2\,q] + \text{Zeta}[-m]\right)}{m}, \{m, 1, \infty\}\right] +$$
$$\text{PoincareSum}\left[\frac{2\,(-1)^{-1+m}\,n^{-m}\,q\left(\left(1 - 2^{-m}\right)\text{HurwitzZeta}[-m, -1+2\,p+2\,q] + \text{Zeta}[-m]\right)}{m}, \{m, 1, \infty\}\right] +$$
$$\text{PolyGamma}[-2, 2\,p] + \text{PolyGamma}[-2, 2\,q]$$

In[ ]:=

$$\mathrm{TMP04} = \frac{5}{4} - 3 p + 2 p^2 - 3 q + 4 p q + 2 q^2 + \frac{13\,\mathrm{Log}[2]}{12} - 2 n\,\mathrm{Log}[2] + n^2\,\mathrm{Log}[2] - 4 p\,\mathrm{Log}[2] + 2 n p\,\mathrm{Log}[2] + 2 p^2\,\mathrm{Log}[2] - 4 q\,\mathrm{Log}[2] + 2 n q\,\mathrm{Log}[2] + 4 p q\,\mathrm{Log}[2] + 2 q^2\,\mathrm{Log}[2] - 3\,\mathrm{Log}[\mathrm{Glaisher}] - \frac{\mathrm{Log}[n]}{4} - n\,\mathrm{Log}[n] + p\,\mathrm{Log}[n] - 2 p^2\,\mathrm{Log}[n] + q\,\mathrm{Log}[n] - 2 q^2\,\mathrm{Log}[n] - p\,\mathrm{Log}[\pi] - q\,\mathrm{Log}[\pi] +$$

$$\left(\mathrm{PoincareSum}\left[\frac{2 (-1)^m n^{-m} \left(\left(1 - 2^{-1-m}\right) \mathrm{HurwitzZeta}[-1 - m, -1 + 2 p + 2 q] + \mathrm{Zeta}[-1 - m]\right)}{1 + m}, \{m, 0, \infty\}\right] \,/.\, \mathrm{PoincareSumSplitOffTerms}[1]\right) +$$

$$\mathrm{PoincareSum}\left[\frac{1}{m} (-1)^m n^{-m} \left(-\frac{\mathrm{HurwitzZeta}[-1 - m, 2 p]}{1 + m} - \frac{\mathrm{HurwitzZeta}[-1 - m, 2 q]}{1 + m} - \frac{\left(1 - 2^{-m} + \left(2 - 2^{-m}\right) m\right) \mathrm{HurwitzZeta}[-1 - m, -1 + 2 p + 2 q]}{1 + m} + 2 p\,\mathrm{HurwitzZeta}[-m, 2 p] + 2 q\,\mathrm{HurwitzZeta}[-m, 2 q] + \left(1 - 2^{-m}\right) (-2 + 2 p + 2 q)\,\mathrm{HurwitzZeta}[-m, -1 + 2 p + 2 q] - \frac{(1 + 2 m)\,\mathrm{Zeta}[-1 - m]}{1 + m} - 2\,\mathrm{Zeta}[-m]\right), \{m, 1, \infty\}\right] +$$

$$\mathrm{PoincareSum}\left[-\frac{2 (-1)^m n^{-m} p\,(\mathrm{HurwitzZeta}[-m, 2 p] - \mathrm{Zeta}[-m])}{m}, \{m, 1, \infty\}\right] +$$

$$\mathrm{PoincareSum}\left[-\frac{2 (-1)^m n^{-m} q\,(\mathrm{HurwitzZeta}[-m, 2 q] - \mathrm{Zeta}[-m])}{m}, \{m, 1, \infty\}\right] +$$

$$\mathrm{PoincareSum}\left[-\frac{2 (-1)^{-1+m} n^{-m} \left(\left(1 - 2^{-m}\right) \mathrm{HurwitzZeta}[-m, -1 + 2 p + 2 q] + \mathrm{Zeta}[-m]\right)}{m}, \{m, 1, \infty\}\right] +$$

$$\mathrm{PoincareSum}\left[\frac{2 (-1)^{-1+m} n^{-m} p \left(\left(1 - 2^{-m}\right) \mathrm{HurwitzZeta}[-m, -1 + 2 p + 2 q] + \mathrm{Zeta}[-m]\right)}{m}, \{m, 1, \infty\}\right] +$$

$$\mathrm{PoincareSum}\left[\frac{2 (-1)^{-1+m} n^{-m} q \left(\left(1 - 2^{-m}\right) \mathrm{HurwitzZeta}[-m, -1 + 2 p + 2 q] + \mathrm{Zeta}[-m]\right)}{m}, \{m, 1, \infty\}\right] +$$

$$\mathrm{PolyGamma}[-2, 2 p] + \mathrm{PolyGamma}[-2, 2 q]$$

Out[ ]=
$$\frac{5}{4} - 3\,p + 2\,p^2 - 3\,q + 4\,p\,q + 2\,q^2 + 2\left(-\frac{1}{12} + \frac{1}{2}\,\text{HurwitzZeta}[-1, -1 + 2\,p + 2\,q]\right) + \frac{13\,\text{Log}[2]}{12} - 2\,n\,\text{Log}[2] + n^2\,\text{Log}[2] - 4\,p\,\text{Log}[2] + 2\,n\,p\,\text{Log}[2] + 2\,p^2\,\text{Log}[2] - 4\,q\,\text{Log}[2] + 2\,n\,q\,\text{Log}[2] + 4\,p\,q\,\text{Log}[2] + 2\,q^2\,\text{Log}[2] - 3\,\text{Log}[\text{Glaisher}] - \frac{\text{Log}[n]}{4} - n\,\text{Log}[n] + p\,\text{Log}[n] - 2\,p^2\,\text{Log}[n] + q\,\text{Log}[n] - 2\,q^2\,\text{Log}[n] - p\,\text{Log}[\pi] - q\,\text{Log}[\pi] +$$

$$\text{PoincareSum}\left[\frac{2\,(-1)^m\,n^{-m}\left(\left(1 - 2^{-1-m}\right)\text{HurwitzZeta}[-1 - m, -1 + 2\,p + 2\,q] + \text{Zeta}[-1 - m]\right)}{1 + m}, \{m, 1, \infty\}\right] +$$

$$\text{PoincareSum}\Bigg[\frac{1}{m}\,(-1)^m\,n^{-m}\Bigg(-\frac{\text{HurwitzZeta}[-1 - m, 2\,p]}{1 + m} - \frac{\text{HurwitzZeta}[-1 - m, 2\,q]}{1 + m} - \frac{\left(1 - 2^{-m} + \left(2 - 2^{-m}\right) m\right)\text{HurwitzZeta}[-1 - m, -1 + 2\,p + 2\,q]}{1 + m} + 2\,p\,\text{HurwitzZeta}[-m, 2\,p] + 2\,q\,\text{HurwitzZeta}[-m, 2\,q] + \left(1 - 2^{-m}\right)(-2 + 2\,p + 2\,q)\,\text{HurwitzZeta}[-m, -1 + 2\,p + 2\,q] - \frac{(1 + 2\,m)\,\text{Zeta}[-1 - m]}{1 + m} - 2\,\text{Zeta}[-m]\Bigg), \{m, 1, \infty\}\Bigg] +$$

$$\text{PoincareSum}\left[-\frac{2\,(-1)^m\,n^{-m}\,p\,(\text{HurwitzZeta}[-m, 2\,p] - \text{Zeta}[-m])}{m}, \{m, 1, \infty\}\right] +$$

$$\text{PoincareSum}\left[-\frac{2\,(-1)^m\,n^{-m}\,q\,(\text{HurwitzZeta}[-m, 2\,q] - \text{Zeta}[-m])}{m}, \{m, 1, \infty\}\right] +$$

$$\text{PoincareSum}\left[-\frac{2\,(-1)^{-1+m}\,n^{-m}\left(\left(1 - 2^{-m}\right)\text{HurwitzZeta}[-m, -1 + 2\,p + 2\,q] + \text{Zeta}[-m]\right)}{m}, \{m, 1, \infty\}\right] +$$

$$\text{PoincareSum}\left[\frac{2\,(-1)^{-1+m}\,n^{-m}\,p\left(\left(1 - 2^{-m}\right)\text{HurwitzZeta}[-m, -1 + 2\,p + 2\,q] + \text{Zeta}[-m]\right)}{m}, \{m, 1, \infty\}\right] +$$

$$\text{PoincareSum}\left[\frac{2\,(-1)^{-1+m}\,n^{-m}\,q\left(\left(1 - 2^{-m}\right)\text{HurwitzZeta}[-m, -1 + 2\,p + 2\,q] + \text{Zeta}[-m]\right)}{m}, \{m, 1, \infty\}\right] +$$

$$\text{PolyGamma}[-2, 2\,p] + \text{PolyGamma}[-2, 2\,q]$$

In[ ]:= `TMP04 //. PoincareSumCollect`

Out[ ]=

$$
\begin{aligned}
&\frac{5}{4} - 3\,p + 2\,p^2 - 3\,q + 4\,p\,q + 2\,q^2 + 2\left(-\frac{1}{12} + \frac{1}{2}\,\text{HurwitzZeta}[-1,\,-1+2\,p+2\,q]\right) + \frac{13\,\text{Log}[2]}{12} - 2\,n\,\text{Log}[2] + \\
&n^2\,\text{Log}[2] - 4\,p\,\text{Log}[2] + 2\,n\,p\,\text{Log}[2] + 2\,p^2\,\text{Log}[2] - 4\,q\,\text{Log}[2] + 2\,n\,q\,\text{Log}[2] + 4\,p\,q\,\text{Log}[2] + 2\,q^2\,\text{Log}[2] - \\
&3\,\text{Log}[\text{Glaisher}] - \frac{\text{Log}[n]}{4} - n\,\text{Log}[n] + p\,\text{Log}[n] - 2\,p^2\,\text{Log}[n] + q\,\text{Log}[n] - 2\,q^2\,\text{Log}[n] - p\,\text{Log}[\pi] - \\
&q\,\text{Log}[\pi] + \text{PoincareSum}\Bigg[\frac{2\,(-1)^m\,n^{-m}\left(\left(1-2^{-1-m}\right)\text{HurwitzZeta}[-1-m,\,-1+2\,p+2\,q] + \text{Zeta}[-1-m]\right)}{1+m} + \\
&\quad \frac{1}{m}\,(-1)^m\,n^{-m}\Bigg(-\frac{\text{HurwitzZeta}[-1-m,\,2\,p]}{1+m} - \frac{\text{HurwitzZeta}[-1-m,\,2\,q]}{1+m} - \\
&\qquad \frac{\left(1-2^{-m}+\left(2-2^{-m}\right)m\right)\text{HurwitzZeta}[-1-m,\,-1+2\,p+2\,q]}{1+m} + \\
&\qquad 2\,p\,\text{HurwitzZeta}[-m,\,2\,p] + 2\,q\,\text{HurwitzZeta}[-m,\,2\,q] + \\
&\qquad \left(1-2^{-m}\right)(-2+2\,p+2\,q)\,\text{HurwitzZeta}[-m,\,-1+2\,p+2\,q] - \frac{(1+2\,m)\,\text{Zeta}[-1-m]}{1+m} - 2\,\text{Zeta}[-m]\Bigg) - \\
&\quad \frac{2\,(-1)^m\,n^{-m}\,p\,(\text{HurwitzZeta}[-m,\,2\,p] - \text{Zeta}[-m])}{m} - \frac{2\,(-1)^m\,n^{-m}\,q\,(\text{HurwitzZeta}[-m,\,2\,q] - \text{Zeta}[-m])}{m} - \\
&\quad \frac{2\,(-1)^{-1+m}\,n^{-m}\left(\left(1-2^{-m}\right)\text{HurwitzZeta}[-m,\,-1+2\,p+2\,q] + \text{Zeta}[-m]\right)}{m} + \\
&\quad \frac{2\,(-1)^{-1+m}\,n^{-m}\,p\left(\left(1-2^{-m}\right)\text{HurwitzZeta}[-m,\,-1+2\,p+2\,q] + \text{Zeta}[-m]\right)}{m} + \\
&\quad \frac{2\,(-1)^{-1+m}\,n^{-m}\,q\left(\left(1-2^{-m}\right)\text{HurwitzZeta}[-m,\,-1+2\,p+2\,q] + \text{Zeta}[-m]\right)}{m}, \\
&\quad \{m,\,1,\,\infty\}\Bigg] + \text{PolyGamma}[-2,\,2\,p] + \text{PolyGamma}[-2,\,2\,q]
\end{aligned}
$$

```
In[∘]:= TMP05 = 5/4 - 3 p + 2 p^2 - 3 q + 4 p q + 2 q^2 + 2 (-1/12 + 1/2 HurwitzZeta[-1, -1 + 2 p + 2 q]) + (13 Log[2])/12 -
   2 n Log[2] + n^2 Log[2] - 4 p Log[2] + 2 n p Log[2] + 2 p^2 Log[2] - 4 q Log[2] + 2 n q Log[2] +
   4 p q Log[2] + 2 q^2 Log[2] - 3 Log[Glaisher] - Log[n]/4 - n Log[n] + p Log[n] - 2 p^2 Log[n] +
   q Log[n] - 2 q^2 Log[n] - p Log[π] - q Log[π] + PolyGamma[-2, 2 p] + PolyGamma[-2, 2 q] +
   PoincareSum[(2 (-1)^m n^-m ((1 - 2^(-1-m)) HurwitzZeta[-1 - m, -1 + 2 p + 2 q] + Zeta[-1 - m]))/(1 + m) +
     1/m (-1)^m n^-m (-HurwitzZeta[-1 - m, 2 p]/(1 + m) - HurwitzZeta[-1 - m, 2 q]/(1 + m) -
       ((1 - 2^-m + (2 - 2^-m) m) HurwitzZeta[-1 - m, -1 + 2 p + 2 q])/(1 + m) + 2 p HurwitzZeta[-m, 2 p] +
       2 q HurwitzZeta[-m, 2 q] + (1 - 2^-m) (-2 + 2 p + 2 q) HurwitzZeta[-m, -1 + 2 p + 2 q] -
       ((1 + 2 m) Zeta[-1 - m])/(1 + m) - 2 Zeta[-m]) - (2 (-1)^m n^-m p (HurwitzZeta[-m, 2 p] - Zeta[-m]))/m -
     (2 (-1)^m n^-m q (HurwitzZeta[-m, 2 q] - Zeta[-m]))/m -
     (2 (-1)^(-1+m) n^-m ((1 - 2^-m) HurwitzZeta[-m, -1 + 2 p + 2 q] + Zeta[-m]))/m +
     (2 (-1)^(-1+m) n^-m p ((1 - 2^-m) HurwitzZeta[-m, -1 + 2 p + 2 q] + Zeta[-m]))/m +
     (2 (-1)^(-1+m) n^-m q ((1 - 2^-m) HurwitzZeta[-m, -1 + 2 p + 2 q] + Zeta[-m]))/m, {m, 1, ∞}];
```

In[ ]:=
$$\text{TMP} = \frac{2\,(-1)^m n^{-m}\left(\left(1-2^{-1-m}\right)\text{HurwitzZeta}[-1-m,\,-1+2p+2q]+\text{Zeta}[-1-m]\right)}{1+m}+$$
$$\frac{1}{m}(-1)^m n^{-m}\left(-\frac{\text{HurwitzZeta}[-1-m,\,2p]}{1+m}-\frac{\text{HurwitzZeta}[-1-m,\,2q]}{1+m}-\frac{\left(1-2^{-m}+\left(2-2^{-m}\right)m\right)\text{HurwitzZeta}[-1-m,\,-1+2p+2q]}{1+m}+\right.$$
$$2p\,\text{HurwitzZeta}[-m,\,2p]+2q\,\text{HurwitzZeta}[-m,\,2q]+$$
$$\left.\left(1-2^{-m}\right)(-2+2p+2q)\,\text{HurwitzZeta}[-m,\,-1+2p+2q]-\frac{(1+2m)\,\text{Zeta}[-1-m]}{1+m}-2\,\text{Zeta}[-m]\right)-$$
$$\frac{2\,(-1)^m n^{-m}\,p\,(\text{HurwitzZeta}[-m,\,2p]-\text{Zeta}[-m])}{m}-\frac{2\,(-1)^m n^{-m}\,q\,(\text{HurwitzZeta}[-m,\,2q]-\text{Zeta}[-m])}{m}-$$
$$\frac{2\,(-1)^{-1+m} n^{-m}\left(\left(1-2^{-m}\right)\text{HurwitzZeta}[-m,\,-1+2p+2q]+\text{Zeta}[-m]\right)}{m}+$$
$$\frac{2\,(-1)^{-1+m} n^{-m}\,p\left(\left(1-2^{-m}\right)\text{HurwitzZeta}[-m,\,-1+2p+2q]+\text{Zeta}[-m]\right)}{m}+$$
$$\frac{2\,(-1)^{-1+m} n^{-m}\,q\left(\left(1-2^{-m}\right)\text{HurwitzZeta}[-m,\,-1+2p+2q]+\text{Zeta}[-m]\right)}{m}\ \text{// Simplify}$$

Out[ ]=
$$\frac{1}{m(1+m)}(-1)^{1+m}2^{-m}n^{-m}\left(2^m\,\text{HurwitzZeta}[-1-m,\,2p]+2^m\,\text{HurwitzZeta}[-1-m,\,2q]-\text{HurwitzZeta}[-1-m,\,-1+2p+2q]+2^m\,\text{HurwitzZeta}[-1-m,\,-1+2p+2q]+2^m\,\text{Zeta}[-1-m]\right)$$

Auxiliary results

In[ ]:=
$$2^{-m}\left(2^m\,\text{HurwitzZeta}[-1-m,\,2p]+2^m\,\text{HurwitzZeta}[-1-m,\,2q]-\text{HurwitzZeta}[-1-m,\,-1+2p+2q]+2^m\,\text{HurwitzZeta}[-1-m,\,-1+2p+2q]+2^m\,\text{Zeta}[-1-m]\right)\ \text{// Expand}$$

Out[ ]=
$$\text{HurwitzZeta}[-1-m,\,2p]+\text{HurwitzZeta}[-1-m,\,2q]+\text{HurwitzZeta}[-1-m,\,-1+2p+2q]-2^{-m}\,\text{HurwitzZeta}[-1-m,\,-1+2p+2q]+\text{Zeta}[-1-m]$$

In[ ]:=
$$+\text{HurwitzZeta}[-1-m,\,-1+2p+2q]-2^{-m}\,\text{HurwitzZeta}[-1-m,\,-1+2p+2q]\ \text{// Simplify}$$

Out[ ]=
$$2^{-m}\left(-1+2^m\right)\text{HurwitzZeta}[-1-m,\,-1+2p+2q]$$

In[ ]:=
$$\left(2^{-m}\left(-1+2^m\right)\ \text{// Distribute}\right)\text{HurwitzZeta}[-1-m,\,-1+2p+2q]$$

Out[ ]=
$$\left(1-2^{-m}\right)\text{HurwitzZeta}[-1-m,\,-1+2p+2q]$$

In[ ]:= TMP == $\frac{(-1)^{m-1}}{m (m+1)}$ (Zeta[-m - 1] + HurwitzZeta[-m - 1, 2 p] + HurwitzZeta[-m - 1, 2 q] + (1 - 2^{-m}) HurwitzZeta[-1 - m, -1 + 2 p + 2 q]) n^{-m} // FullSimplify

Out[ ]= True

In[ ]:= TMP06 = $\frac{5}{4}$ - 3 p + 2 p^2 - 3 q + 4 p q + 2 q^2 + 2 (- $\frac{1}{12}$ + $\frac{1}{2}$ HurwitzZeta[-1, -1 + 2 p + 2 q]) + $\frac{13\,\text{Log}[2]}{12}$ - 2 n Log[2] + n^2 Log[2] - 4 p Log[2] + 2 n p Log[2] + 2 p^2 Log[2] - 4 q Log[2] + 2 n q Log[2] + 4 p q Log[2] + 2 q^2 Log[2] - 3 Log[Glaisher] - $\frac{\text{Log}[n]}{4}$ - n Log[n] + p Log[n] - 2 p^2 Log[n] + q Log[n] - 2 q^2 Log[n] - p Log[π] - q Log[π] + PolyGamma[-2, 2 p] + PolyGamma[-2, 2 q] + PoincareSum[$\frac{(-1)^{m-1}}{m (m+1)}$ (Zeta[-m - 1] + HurwitzZeta[-m - 1, 2 p] + HurwitzZeta[-m - 1, 2 q] + (1 - 2^{-m}) HurwitzZeta[-1 - m, -1 + 2 p + 2 q]) n^{-m}, {m, 1, ∞}];

Leading terms

In[ ]:= TMP = $\frac{5}{4}$ - 3 p + 2 p^2 - 3 q + 4 p q + 2 q^2 + 2 (- $\frac{1}{12}$ + $\frac{1}{2}$ HurwitzZeta[-1, -1 + 2 p + 2 q]) + $\frac{13\,\text{Log}[2]}{12}$ - 2 n Log[2] + n^2 Log[2] - 4 p Log[2] + 2 n p Log[2] + 2 p^2 Log[2] - 4 q Log[2] + 2 n q Log[2] + 4 p q Log[2] + 2 q^2 Log[2] - 3 Log[Glaisher] - $\frac{\text{Log}[n]}{4}$ - n Log[n] + p Log[n] - 2 p^2 Log[n] + q Log[n] - 2 q^2 Log[n] - p Log[π] - q Log[π] + PolyGamma[-2, 2 p] + PolyGamma[-2, 2 q];

Auxiliary results

In[ ]:= $\frac{5}{4} - 3p + 2p^2 - 3q + 4pq + 2q^2 + 2\left(-\frac{1}{12} + \frac{1}{2}\text{HurwitzZeta}[-1, -1 + 2p + 2q]\right) + \frac{13\,\text{Log}[2]}{12} - 2n\,\text{Log}[2] + n^2\,\text{Log}[2] - 4p\,\text{Log}[2] + 2np\,\text{Log}[2] + 2p^2\,\text{Log}[2] - 4q\,\text{Log}[2] + 2nq\,\text{Log}[2] + 4pq\,\text{Log}[2] + 2q^2\,\text{Log}[2] - 3\,\text{Log}[\text{Glaisher}] - \frac{\text{Log}[n]}{4} - n\,\text{Log}[n] + p\,\text{Log}[n] - 2p^2\,\text{Log}[n] + q\,\text{Log}[n] - 2q^2\,\text{Log}[n] - p\,\text{Log}[\pi] - q\,\text{Log}[\pi] + \text{PolyGamma}[-2, 2p] + \text{PolyGamma}[-2, 2q]$ // FunctionExpand // Expand

Out[ ]= $\frac{13\,\text{Log}[2]}{12} - 2n\,\text{Log}[2] + n^2\,\text{Log}[2] - 4p\,\text{Log}[2] + 2np\,\text{Log}[2] + 2p^2\,\text{Log}[2] - 4q\,\text{Log}[2] + 2nq\,\text{Log}[2] + 4pq\,\text{Log}[2] + 2q^2\,\text{Log}[2] - 3\,\text{Log}[\text{Glaisher}] - \frac{\text{Log}[n]}{4} - n\,\text{Log}[n] + p\,\text{Log}[n] - 2p^2\,\text{Log}[n] + q\,\text{Log}[n] - 2q^2\,\text{Log}[n] - p\,\text{Log}[\pi] - q\,\text{Log}[\pi] + \text{PolyGamma}[-2, 2p] + \text{PolyGamma}[-2, 2q]$

In[ ]:= $-2n\,\text{Log}[2] + 2np\,\text{Log}[2] + 2nq\,\text{Log}[2]$ // Simplify

Out[ ]= $2n(-1 + p + q)\,\text{Log}[2]$

In[ ]:= $\frac{13\,\text{Log}[2]}{12} - 4p\,\text{Log}[2] + 2p^2\,\text{Log}[2] - 4q\,\text{Log}[2] + 4pq\,\text{Log}[2] + 2q^2\,\text{Log}[2] - 3\,\text{Log}[\text{Glaisher}] - \frac{\text{Log}[n]}{4} + p\,\text{Log}[n] - 2p^2\,\text{Log}[n] + q\,\text{Log}[n] - 2q^2\,\text{Log}[n] - p\,\text{Log}[\pi] - q\,\text{Log}[\pi] + \text{PolyGamma}[-2, 2p] + \text{PolyGamma}[-2, 2q]$ // Collect[#, {Log[n]}] &

Out[ ]= $\frac{13\,\text{Log}[2]}{12} - 4p\,\text{Log}[2] + 2p^2\,\text{Log}[2] - 4q\,\text{Log}[2] + 4pq\,\text{Log}[2] + 2q^2\,\text{Log}[2] - 3\,\text{Log}[\text{Glaisher}] + \left(-\frac{1}{4} + p - 2p^2 + q - 2q^2\right)\text{Log}[n] - p\,\text{Log}[\pi] - q\,\text{Log}[\pi] + \text{PolyGamma}[-2, 2p] + \text{PolyGamma}[-2, 2q]$

In[ ]:= $+\left(-\frac{1}{4} + p - 2p^2 + q - 2q^2\right) == -2\left(\left(p - \frac{1}{4}\right)^2 + \left(q - \frac{1}{4}\right)^2\right)$ // FullSimplify

Out[ ]= True

In[ ]:= $\frac{13\,\text{Log}[2]}{12} - 4p\,\text{Log}[2] + 2p^2\,\text{Log}[2] - 4q\,\text{Log}[2] + 4pq\,\text{Log}[2] + 2q^2\,\text{Log}[2] - 3\,\text{Log}[\text{Glaisher}] - p\,\text{Log}[\pi] - q\,\text{Log}[\pi]$ // Collect[#, {Log[2], Log[π]}] &

Out[ ]= $\left(\frac{13}{12} - 4p + 2p^2 - 4q + 4pq + 2q^2\right)\text{Log}[2] - 3\,\text{Log}[\text{Glaisher}] + (-p - q)\,\text{Log}[\pi]$

In[ ]:= $\frac{13}{12} - 4p + 2p^2 - 4q + 4pq + 2q^2 == 2\left((p + q - 1)^2 - \frac{11}{24}\right)$ // FullSimplify

Out[ ]= True

In[ ]:=
```
TMP == Log[2] n^2 - n Log[n] + 2 Log[2] (p + q - 1) n -
   2 ((p - 1/4)^2 + (q - 1/4)^2) Log[n] + 2 ((p + q - 1)^2 - 11/24) Log[2] - (p + q) Log[π] -
   3 Log[Glaisher] + PolyGamma[-2, 2 p] + PolyGamma[-2, 2 q] // FullSimplify
```

Out[ ]= True

```
TMP07 = Log[2] n^2 - n Log[n] + 2 Log[2] (p + q - 1) n -
   2 ((p - 1/4)^2 + (q - 1/4)^2) Log[n] + 2 ((p + q - 1)^2 - 11/24) Log[2] - (p + q) Log[π] -
   3 Log[Glaisher] + PolyGamma[-2, 2 p] + PolyGamma[-2, 2 q] +
   PoincareSum[(-1)^(m-1)/(m (m + 1)) (Zeta[-m - 1] + HurwitzZeta[-m - 1, 2 p] + HurwitzZeta[-m - 1, 2 q] +
       (1 - 2^-m) HurwitzZeta[-1 - m, -1 + 2 p + 2 q]) n^-m, {m, 1, ∞}];
```

Formula

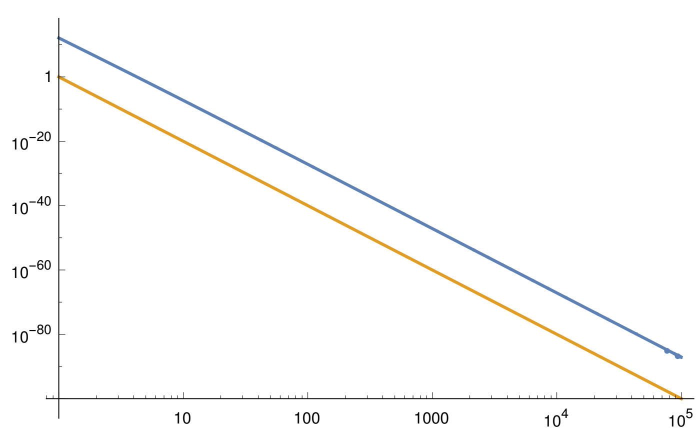
In[ ]:=
```
ASYMPL[n_, q_, p_] := Log[2] n^2 - n Log[n] +
   2 Log[2] (p + q - 1) n - 2 ((p - 1/4)^2 + (q - 1/4)^2) Log[n] + 2 ((p + q - 1)^2 - 11/24) Log[2] -
   (p + q) Log[π] - 3 Log[Glaisher] + PolyGamma[-2, 2 p] + PolyGamma[-2, 2 q] +
   PoincareSum[(-1)^(m-1)/(m (m + 1)) (Zeta[-m - 1] + HurwitzZeta[-m - 1, 2 p] + HurwitzZeta[-m - 1, 2 q] +
       (1 - 2^-m) HurwitzZeta[-1 - m, -1 + 2 p + 2 q]) n^-m, {m, 1, ∞}];
```

Cross Check

```
2 (n + p + q - 1) Logλ[n, α, β] - LogD[n, α, β] - 2 p LogP[n, α, β] - 2 q LogP[n, β, α];
(* for comparison *)
```

In[ ]:=
```
λ[n_, α_, β_] := 2^-n Binomial[2 n + α + β, n];
Discr[n_, α_, β_] := 2^(-n (n-1)) Product[ν^(ν-2 n+2) (ν + α)^(ν-1) (ν + β)^(ν-1) (ν + n + α + β)^(n-ν), {ν, 1, n}];

n =.; (*   *)

p = 1 / 2;
q = π;

α = 2 p - 1;
β = 2 q - 1;


REFLogλ = - Log[2] n + Log[Gamma[2 n + α + β + 1]] - Log[Gamma[n + α + β + 1]] - Log[Gamma[n + 1]];
REFLogD = - n (n - 1) Log[2] + FracA2[n] + FracB2[n, α] + FracB2[n, β] + FracC2[n, α + β];
REFLogP[n_, α_, β_] := - Log[Gamma[α + 1]] + Log[Gamma[n + α + 1]] - Log[Gamma[n + 1]];

REF = 2 (n + p + q - 1) REFLogλ - REFLogD - 2 p REFLogP[n, α, β] - 2 q REFLogP[n, β, α];

(* -Log[(JacobiP[n,α,β,1]^p/λ[n,α,β]^p Sqrt[Discr[n,α,β]]/λ[n,α,β]^(n-1) ((-1)^n JacobiP[n,α,β,-1])^q/λ[n,α,β]^q)^2] *)


K = 20;

ASYMP = ASYMPL[n, q, p] /. PoincareSumNormalize[K - 1];

LogLogPlot[{Abs[REF - ASYMP], n^-K}, {n, 1, 100 000}, WorkingPrecision → 512]

Clear[α, β, n, p, q, zeros, K];
```

Out[ ]=



Proof of Theorem 1.3

Verification of starting point

Formula clear by definition of logarithmic energy and definition of discriminant.

Application of asymptotics

```
2 (n - 1) Logλ[n, α, β] - LogD[n, α, β];

ASYMPλ[n, α, β];
ASYMPD[n, α, β];

(* for comparison ... *)
```

In[ ]:=
```
α = 2 p - 1;
β = 2 q - 1;

TMP01 = 2 (n - 1) ASYMPλ[n, α, β] - ASYMPD[n, α, β] // Expand

Clear[α, β];
```

Out[ ]=

$$\frac{5}{4} - 3 p + 2 p^2 - 3 q + 4 p q + 2 q^2 + \frac{13 \text{Log}[2]}{12} - 2 n \text{Log}[2] + n^2 \text{Log}[2] - 2 p^2 \text{Log}[2] - 4 p q \text{Log}[2] - 2 q^2 \text{Log}[2] - 3 \text{Log}[\text{Glaisher}] - \frac{\text{Log}[n]}{4} - n \text{Log}[n] + 2 p^2 \text{Log}[n] + 2 q^2 \text{Log}[n] - 2 p \text{Log}[\text{Gamma}[2 p]] - 2 q \text{Log}[\text{Gamma}[2 q]] -$$

$$\text{PoincareSum}\Big[\frac{1}{m} (-1)^{-1+m} n^{-m} \Big(-\frac{\text{HurwitzZeta}[-1-m, 2 p]}{1+m} - \frac{\text{HurwitzZeta}[-1-m, 2 q]}{1+m} - \frac{(1 - 2^{-m} + (2 - 2^{-m}) m) \text{HurwitzZeta}[-1-m, -1+2 p+2 q]}{1+m} + 2 p \text{HurwitzZeta}[-m, 2 p] + 2 q \text{HurwitzZeta}[-m, 2 q] + (1 - 2^{-m}) (-2 + 2 p + 2 q) \text{HurwitzZeta}[-m, -1+2 p+2 q] - \frac{(1+2 m) \text{Zeta}[-1-m]}{1+m} - 2 \text{Zeta}[-m]\Big), \{m, 1, \infty\}\Big] -$$

$$2 \text{PoincareSum}\Big[\frac{(-1)^{-1+m} n^{-m} ((1 - 2^{-m}) \text{HurwitzZeta}[-m, -1+2 p+2 q] + \text{Zeta}[-m])}{m}, \{m, 1, \infty\}\Big] +$$

$$2 n \text{PoincareSum}\Big[\frac{(-1)^{-1+m} n^{-m} ((1 - 2^{-m}) \text{HurwitzZeta}[-m, -1+2 p+2 q] + \text{Zeta}[-m])}{m}, \{m, 1, \infty\}\Big] +$$

$$\text{PolyGamma}[-2, 2 p] + \text{PolyGamma}[-2, 2 q]$$

Simplification

In[ ]:= `TMP01 /. PoincareSumFactorUnderSum`

Out[ ]=
$$\frac{5}{4} - 3 p + 2 p^2 - 3 q + 4 p q + 2 q^2 + \frac{13 \text{Log}[2]}{12} - 2 n \text{Log}[2] + n^2 \text{Log}[2] - 2 p^2 \text{Log}[2] - 4 p q \text{Log}[2] - 2 q^2 \text{Log}[2] - 3 \text{Log}[\text{Glaisher}] - \frac{\text{Log}[n]}{4} - n \text{Log}[n] + 2 p^2 \text{Log}[n] + 2 q^2 \text{Log}[n] - 2 p \text{Log}[\text{Gamma}[2 p]] - 2 q \text{Log}[\text{Gamma}[2 q]] +$$
$$\text{PoincareSum}\Big[\frac{1}{m} (-1)^m n^{-m} \Big(-\frac{\text{HurwitzZeta}[-1 - m, 2 p]}{1 + m} - \frac{\text{HurwitzZeta}[-1 - m, 2 q]}{1 + m} - \frac{\left(1 - 2^{-m} + \left(2 - 2^{-m}\right) m\right) \text{HurwitzZeta}[-1 - m, -1 + 2 p + 2 q]}{1 + m} + 2 p \text{HurwitzZeta}[-m, 2 p] + 2 q \text{HurwitzZeta}[-m, 2 q] + \left(1 - 2^{-m}\right) (-2 + 2 p + 2 q) \text{HurwitzZeta}[-m, -1 + 2 p + 2 q] - \frac{(1 + 2 m) \text{Zeta}[-1 - m]}{1 + m} - 2 \text{Zeta}[-m]\Big), \{m, 1, \infty\}\Big] +$$
$$\text{PoincareSum}\Big[\frac{2 (-1)^{-1+m} n^{1-m} \left(\left(1 - 2^{-m}\right) \text{HurwitzZeta}[-m, -1 + 2 p + 2 q] + \text{Zeta}[-m]\right)}{m}, \{m, 1, \infty\}\Big] +$$
$$\text{PoincareSum}\Big[-\frac{2 (-1)^{-1+m} n^{-m} \left(\left(1 - 2^{-m}\right) \text{HurwitzZeta}[-m, -1 + 2 p + 2 q] + \text{Zeta}[-m]\right)}{m}, \{m, 1, \infty\}\Big] +$$
$$\text{PolyGamma}[-2, 2 p] + \text{PolyGamma}[-2, 2 q]$$

In[ ]:= TMP02 = $\frac{5}{4} - 3 p + 2 p^2 - 3 q + 4 p q + 2 q^2 + \frac{13 \text{Log}[2]}{12} - 2 n \text{Log}[2] + n^2 \text{Log}[2] - 2 p^2 \text{Log}[2] - 4 p q \text{Log}[2] - 2 q^2 \text{Log}[2] - 3 \text{Log}[\text{Glaisher}] - \frac{\text{Log}[n]}{4} - n \text{Log}[n] + 2 p^2 \text{Log}[n] + 2 q^2 \text{Log}[n] - 2 p \text{Log}[\text{Gamma}[2 p]] - 2 q \text{Log}[\text{Gamma}[2 q]] +$

$\text{PoincareSum}\Big[\frac{1}{m} (-1)^m n^{-m} \Big(-\frac{\text{HurwitzZeta}[-1-m, 2 p]}{1+m} - \frac{\text{HurwitzZeta}[-1-m, 2 q]}{1+m} - \frac{(1 - 2^{-m} + (2 - 2^{-m}) m) \text{HurwitzZeta}[-1-m, -1+2 p+2 q]}{1+m} + 2 p \text{HurwitzZeta}[-m, 2 p] + 2 q \text{HurwitzZeta}[-m, 2 q] + (1 - 2^{-m}) (-2 + 2 p + 2 q) \text{HurwitzZeta}[-m, -1+2 p+2 q] - \frac{(1+2 m) \text{Zeta}[-1-m]}{1+m} - 2 \text{Zeta}[-m]\Big), \{m, 1, \infty\}\Big] +$

$\Big(\text{PoincareSum}\Big[\frac{2 (-1)^{-1+m} n^{1-m} ((1 - 2^{-m}) \text{HurwitzZeta}[-m, -1+2 p+2 q] + \text{Zeta}[-m])}{m}, \{m, 1, \infty\}\Big] \text{/.} \text{PoincareSumIndexShiftUp}[1]\Big) +$

$\text{PoincareSum}\Big[-\frac{2 (-1)^{-1+m} n^{-m} ((1 - 2^{-m}) \text{HurwitzZeta}[-m, -1+2 p+2 q] + \text{Zeta}[-m])}{m}, \{m, 1, \infty\}\Big] +$

$\text{PolyGamma}[-2, 2 p] + \text{PolyGamma}[-2, 2 q]$

Out[ ]=
$$\frac{5}{4} - 3 p + 2 p^2 - 3 q + 4 p q + 2 q^2 + \frac{13 \text{Log}[2]}{12} - 2 n \text{Log}[2] + n^2 \text{Log}[2] - 2 p^2 \text{Log}[2] - 4 p q \text{Log}[2] - 2 q^2 \text{Log}[2] - 3 \text{Log}[\text{Glaisher}] - \frac{\text{Log}[n]}{4} - n \text{Log}[n] + 2 p^2 \text{Log}[n] + 2 q^2 \text{Log}[n] - 2 p \text{Log}[\text{Gamma}[2 p]] - 2 q \text{Log}[\text{Gamma}[2 q]] +$$
$$\text{PoincareSum}\left[\frac{2 (-1)^m n^{-m} \left(\left(1 - 2^{-1-m}\right) \text{HurwitzZeta}[-1 - m, -1 + 2 p + 2 q] + \text{Zeta}[-1 - m]\right)}{1 + m}, \{m, 0, \infty\}\right] +$$
$$\text{PoincareSum}\left[\frac{1}{m} (-1)^m n^{-m} \left(-\frac{\text{HurwitzZeta}[-1 - m, 2 p]}{1 + m} - \frac{\text{HurwitzZeta}[-1 - m, 2 q]}{1 + m} - \frac{\left(1 - 2^{-m} + \left(2 - 2^{-m}\right) m\right) \text{HurwitzZeta}[-1 - m, -1 + 2 p + 2 q]}{1 + m} + 2 p \text{HurwitzZeta}[-m, 2 p] + 2 q \text{HurwitzZeta}[-m, 2 q] + \left(1 - 2^{-m}\right) (-2 + 2 p + 2 q) \text{HurwitzZeta}[-m, -1 + 2 p + 2 q] - \frac{(1 + 2 m) \text{Zeta}[-1 - m]}{1 + m} - 2 \text{Zeta}[-m]\right), \{m, 1, \infty\}\right] +$$
$$\text{PoincareSum}\left[-\frac{2 (-1)^{-1+m} n^{-m} \left(\left(1 - 2^{-m}\right) \text{HurwitzZeta}[-m, -1 + 2 p + 2 q] + \text{Zeta}[-m]\right)}{m}, \{m, 1, \infty\}\right] +$$
$$\text{PolyGamma}[-2, 2 p] + \text{PolyGamma}[-2, 2 q]$$

In[ ]:= TMP03 = $\frac{5}{4} - 3\,p + 2\,p^2 - 3\,q + 4\,p\,q + 2\,q^2 + \frac{13\,\text{Log}[2]}{12} - 2\,n\,\text{Log}[2] +$

$n^2\,\text{Log}[2] - 2\,p^2\,\text{Log}[2] - 4\,p\,q\,\text{Log}[2] - 2\,q^2\,\text{Log}[2] - 3\,\text{Log}[\text{Glaisher}] - \frac{\text{Log}[n]}{4} -$

$n\,\text{Log}[n] + 2\,p^2\,\text{Log}[n] + 2\,q^2\,\text{Log}[n] - 2\,p\,\text{Log}[\text{Gamma}[2\,p]] - 2\,q\,\text{Log}[\text{Gamma}[2\,q]] +$

$\left(\text{PoincareSum}\left[\frac{2\,(-1)^m\,n^{-m}\,\left(\left(1 - 2^{-1-m}\right)\,\text{HurwitzZeta}[-1-m,\,-1+2\,p+2\,q] + \text{Zeta}[-1-m]\right)}{1+m},\right.\right.$

$\left.\left.\{m,\,0,\,\infty\}\right]\ /.\ \text{PoincareSumSplitOffTerms}[1]\right) +$

$\text{PoincareSum}\left[\frac{1}{m}\,(-1)^m\,n^{-m}\left(-\frac{\text{HurwitzZeta}[-1-m,\,2\,p]}{1+m} - \frac{\text{HurwitzZeta}[-1-m,\,2\,q]}{1+m} -\right.\right.$

$\frac{\left(1 - 2^{-m} + \left(2 - 2^{-m}\right)\,m\right)\,\text{HurwitzZeta}[-1-m,\,-1+2\,p+2\,q]}{1+m} + 2\,p\,\text{HurwitzZeta}[-m,\,2\,p] +$

$2\,q\,\text{HurwitzZeta}[-m,\,2\,q] + \left(1 - 2^{-m}\right)\,(-2+2\,p+2\,q)\,\text{HurwitzZeta}[-m,\,-1+2\,p+2\,q] -$

$\left.\left.\frac{(1+2\,m)\,\text{Zeta}[-1-m]}{1+m} - 2\,\text{Zeta}[-m]\right),\,\{m,\,1,\,\infty\}\right] +$

$\text{PoincareSum}\left[-\frac{2\,(-1)^{-1+m}\,n^{-m}\,\left(\left(1 - 2^{-m}\right)\,\text{HurwitzZeta}[-m,\,-1+2\,p+2\,q] + \text{Zeta}[-m]\right)}{m},\,\{m,\,1,\,\infty\}\right] +$

$\text{PolyGamma}[-2,\,2\,p] + \text{PolyGamma}[-2,\,2\,q]$

Out[ ]=
$$\frac{5}{4} - 3\,p + 2\,p^2 - 3\,q + 4\,p\,q + 2\,q^2 + 2\left(-\frac{1}{12} + \frac{1}{2}\,\text{HurwitzZeta}[-1, -1+2\,p+2\,q]\right) + \frac{13\,\text{Log}[2]}{12} - 2\,n\,\text{Log}[2] + n^2\,\text{Log}[2] - 2\,p^2\,\text{Log}[2] - 4\,p\,q\,\text{Log}[2] - 2\,q^2\,\text{Log}[2] - 3\,\text{Log}[\text{Glaisher}] - \frac{\text{Log}[n]}{4} - n\,\text{Log}[n] + 2\,p^2\,\text{Log}[n] + 2\,q^2\,\text{Log}[n] - 2\,p\,\text{Log}[\text{Gamma}[2\,p]] - 2\,q\,\text{Log}[\text{Gamma}[2\,q]] + \text{PoincareSum}\left[\frac{2\,(-1)^m\,n^{-m}\left(\left(1-2^{-1-m}\right)\text{HurwitzZeta}[-1-m, -1+2\,p+2\,q] + \text{Zeta}[-1-m]\right)}{1+m}, \{m, 1, \infty\}\right] + \text{PoincareSum}\left[\frac{1}{m}\,(-1)^m\,n^{-m}\left(-\frac{\text{HurwitzZeta}[-1-m, 2\,p]}{1+m} - \frac{\text{HurwitzZeta}[-1-m, 2\,q]}{1+m} - \frac{\left(1-2^{-m}+\left(2-2^{-m}\right)m\right)\text{HurwitzZeta}[-1-m, -1+2\,p+2\,q]}{1+m} + 2\,p\,\text{HurwitzZeta}[-m, 2\,p] + 2\,q\,\text{HurwitzZeta}[-m, 2\,q] + \left(1-2^{-m}\right)(-2+2\,p+2\,q)\,\text{HurwitzZeta}[-m, -1+2\,p+2\,q] - \frac{(1+2\,m)\,\text{Zeta}[-1-m]}{1+m} - 2\,\text{Zeta}[-m]\right), \{m, 1, \infty\}\right] + \text{PoincareSum}\left[-\frac{2\,(-1)^{-1+m}\,n^{-m}\left(\left(1-2^{-m}\right)\text{HurwitzZeta}[-m, -1+2\,p+2\,q] + \text{Zeta}[-m]\right)}{m}, \{m, 1, \infty\}\right] + \text{PolyGamma}[-2, 2\,p] + \text{PolyGamma}[-2, 2\,q]$$

In[ ]:= **TMP03 //. PoincareSumCollect**

Out[ ]=
$$\frac{5}{4} - 3\,p + 2\,p^2 - 3\,q + 4\,p\,q + 2\,q^2 + 2\left(-\frac{1}{12} + \frac{1}{2}\,\text{HurwitzZeta}[-1, -1+2\,p+2\,q]\right) + \frac{13\,\text{Log}[2]}{12} - 2\,n\,\text{Log}[2] + n^2\,\text{Log}[2] - 2\,p^2\,\text{Log}[2] - 4\,p\,q\,\text{Log}[2] - 2\,q^2\,\text{Log}[2] - 3\,\text{Log}[\text{Glaisher}] - \frac{\text{Log}[n]}{4} - n\,\text{Log}[n] + 2\,p^2\,\text{Log}[n] + 2\,q^2\,\text{Log}[n] - 2\,p\,\text{Log}[\text{Gamma}[2\,p]] - 2\,q\,\text{Log}[\text{Gamma}[2\,q]] + \text{PoincareSum}\left[\frac{2\,(-1)^m\,n^{-m}\left(\left(1-2^{-1-m}\right)\text{HurwitzZeta}[-1-m, -1+2\,p+2\,q] + \text{Zeta}[-1-m]\right)}{1+m} + \frac{1}{m}\,(-1)^m\,n^{-m}\left(-\frac{\text{HurwitzZeta}[-1-m, 2\,p]}{1+m} - \frac{\text{HurwitzZeta}[-1-m, 2\,q]}{1+m} - \frac{\left(1-2^{-m}+\left(2-2^{-m}\right)m\right)\text{HurwitzZeta}[-1-m, -1+2\,p+2\,q]}{1+m} + 2\,p\,\text{HurwitzZeta}[-m, 2\,p] + 2\,q\,\text{HurwitzZeta}[-m, 2\,q] + \left(1-2^{-m}\right)(-2+2\,p+2\,q)\,\text{HurwitzZeta}[-m, -1+2\,p+2\,q] - \frac{(1+2\,m)\,\text{Zeta}[-1-m]}{1+m} - 2\,\text{Zeta}[-m]\right) - \frac{2\,(-1)^{-1+m}\,n^{-m}\left(\left(1-2^{-m}\right)\text{HurwitzZeta}[-m, -1+2\,p+2\,q] + \text{Zeta}[-m]\right)}{m}, \{m, 1, \infty\}\right] + \text{PolyGamma}[-2, 2\,p] + \text{PolyGamma}[-2, 2\,q]$$

In[ ]:=
$$\text{TMP04} = \frac{5}{4} - 3\,p + 2\,p^2 - 3\,q + 4\,p\,q + 2\,q^2 + 2\left(-\frac{1}{12} + \frac{1}{2}\,\text{HurwitzZeta}[-1, -1 + 2\,p + 2\,q]\right) + \frac{13\,\text{Log}[2]}{12} - 2\,n\,\text{Log}[2] + n^2\,\text{Log}[2] - 2\,p^2\,\text{Log}[2] - 4\,p\,q\,\text{Log}[2] - 2\,q^2\,\text{Log}[2] - 3\,\text{Log}[\text{Glaisher}] - \frac{\text{Log}[n]}{4} - n\,\text{Log}[n] + 2\,p^2\,\text{Log}[n] + 2\,q^2\,\text{Log}[n] - 2\,p\,\text{Log}[\text{Gamma}[2\,p]] - 2\,q\,\text{Log}[\text{Gamma}[2\,q]] + \text{PolyGamma}[-2, 2\,p] + \text{PolyGamma}[-2, 2\,q] + \text{PoincareSum}\Big[\frac{2\,(-1)^m\,n^{-m}\left(\left(1 - 2^{-1-m}\right)\text{HurwitzZeta}[-1-m, -1+2\,p+2\,q] + \text{Zeta}[-1-m]\right)}{1+m} + \frac{1}{m}\,(-1)^m\,n^{-m}\left(-\frac{\text{HurwitzZeta}[-1-m, 2\,p]}{1+m} - \frac{\text{HurwitzZeta}[-1-m, 2\,q]}{1+m} - \frac{\left(1 - 2^{-m} + \left(2 - 2^{-m}\right)m\right)\text{HurwitzZeta}[-1-m, -1+2\,p+2\,q]}{1+m} + 2\,p\,\text{HurwitzZeta}[-m, 2\,p] + 2\,q\,\text{HurwitzZeta}[-m, 2\,q] + \left(1 - 2^{-m}\right)(-2+2\,p+2\,q)\,\text{HurwitzZeta}[-m, -1+2\,p+2\,q] - \frac{(1+2\,m)\,\text{Zeta}[-1-m]}{1+m} - 2\,\text{Zeta}[-m]\right) - \frac{2\,(-1)^{-1+m}\,n^{-m}\left(\left(1 - 2^{-m}\right)\text{HurwitzZeta}[-m, -1+2\,p+2\,q] + \text{Zeta}[-m]\right)}{m}, \{m, 1, \infty\}\Big];$$

In[ ]:=
```
TMP = (2 (-1)^m n^-m ((1 - 2^(-1-m)) HurwitzZeta[-1 - m, -1 + 2 p + 2 q] + Zeta[-1 - m]))/(1 + m) +
  1/m (-1)^m n^-m (-HurwitzZeta[-1 - m, 2 p]/(1 + m) - HurwitzZeta[-1 - m, 2 q]/(1 + m) -
     ((1 - 2^-m + (2 - 2^-m) m) HurwitzZeta[-1 - m, -1 + 2 p + 2 q])/(1 + m) +
     2 p HurwitzZeta[-m, 2 p] + 2 q HurwitzZeta[-m, 2 q] +
     (1 - 2^-m) (-2 + 2 p + 2 q) HurwitzZeta[-m, -1 + 2 p + 2 q] - ((1 + 2 m) Zeta[-1 - m])/(1 + m) - 2 Zeta[-m]) -
  (2 (-1)^(-1+m) n^-m ((1 - 2^-m) HurwitzZeta[-m, -1 + 2 p + 2 q] + Zeta[-m]))/m // Simplify
```

Out[ ]=
```
1/(m (1 + m)) (-1)^(1+m) n^-m
 (HurwitzZeta[-1 - m, 2 p] + HurwitzZeta[-1 - m, 2 q] + HurwitzZeta[-1 - m, -1 + 2 p + 2 q] -
   2^-m HurwitzZeta[-1 - m, -1 + 2 p + 2 q] - 2 (1 + m) p HurwitzZeta[-m, 2 p] -
   2 (1 + m) q HurwitzZeta[-m, 2 q] - 2 (1 + m) p HurwitzZeta[-m, -1 + 2 p + 2 q] +
   2^(1-m) (1 + m) p HurwitzZeta[-m, -1 + 2 p + 2 q] - 2 (1 + m) q HurwitzZeta[-m, -1 + 2 p + 2 q] +
   2^(1-m) (1 + m) q HurwitzZeta[-m, -1 + 2 p + 2 q] + Zeta[-1 - m])
```

Auxiliary results

In[ ]:=
```
1/(1 + m) (HurwitzZeta[-1 - m, 2 p] + HurwitzZeta[-1 - m, 2 q] + HurwitzZeta[-1 - m, -1 + 2 p + 2 q] -
   2^-m HurwitzZeta[-1 - m, -1 + 2 p + 2 q] - 2 (1 + m) p HurwitzZeta[-m, 2 p] -
   2 (1 + m) q HurwitzZeta[-m, 2 q] - 2 (1 + m) p HurwitzZeta[-m, -1 + 2 p + 2 q] +
   2^(1-m) (1 + m) p HurwitzZeta[-m, -1 + 2 p + 2 q] - 2 (1 + m) q HurwitzZeta[-m, -1 + 2 p + 2 q] +
   2^(1-m) (1 + m) q HurwitzZeta[-m, -1 + 2 p + 2 q] + Zeta[-1 - m]) // Expand
```

In[ ]:=
$$\frac{\text{HurwitzZeta}[-1-m, 2p]}{1+m} + \frac{\text{HurwitzZeta}[-1-m, 2q]}{1+m} + \frac{\text{HurwitzZeta}[-1-m, -1+2p+2q]}{1+m} - \frac{2^{-m}\,\text{HurwitzZeta}[-1-m, -1+2p+2q]}{1+m} - \frac{2p\,\text{HurwitzZeta}[-m, 2p]}{1+m} - \frac{2mp\,\text{HurwitzZeta}[-m, 2p]}{1+m} - \frac{2q\,\text{HurwitzZeta}[-m, 2q]}{1+m} - \frac{2mq\,\text{HurwitzZeta}[-m, 2q]}{1+m} - \frac{2p\,\text{HurwitzZeta}[-m, -1+2p+2q]}{1+m} + \frac{2^{1-m}p\,\text{HurwitzZeta}[-m, -1+2p+2q]}{1+m} - \frac{2mp\,\text{HurwitzZeta}[-m, -1+2p+2q]}{1+m} + \frac{2^{1-m}mp\,\text{HurwitzZeta}[-m, -1+2p+2q]}{1+m} - \frac{2q\,\text{HurwitzZeta}[-m, -1+2p+2q]}{1+m} + \frac{2^{1-m}q\,\text{HurwitzZeta}[-m, -1+2p+2q]}{1+m} - \frac{2mq\,\text{HurwitzZeta}[-m, -1+2p+2q]}{1+m} + \frac{2^{1-m}mq\,\text{HurwitzZeta}[-m, -1+2p+2q]}{1+m} + \frac{\text{Zeta}[-1-m]}{1+m} \text{ // Collect}\left[\#, \left\{\frac{a\_}{m+1}\right\}, \text{Simplify}\right]\&$$

In[ ]:=
$$-2^{1-m}\left(-1+2^{m}\right)p\,\text{HurwitzZeta}[-m, -1+2p+2q] - 2^{1-m}\left(-1+2^{m}\right)q\,\text{HurwitzZeta}[-m, -1+2p+2q] \text{ // FullSimplify}$$

In[ ]:=
$$-2\left(2^{-m}\left(-1+2^{m}\right) \text{ // Expand}\right)(p+q)\,\text{HurwitzZeta}[-m, -1+2p+2q]$$

Out[ ]=
$$-2\left(1-2^{-m}\right)(p+q)\,\text{HurwitzZeta}[-m, -1+2p+2q]$$

In[ ]:=
$$\text{TMP} == \frac{(-1)^{m-1}}{m}\left(\frac{1}{m+1}\left(\text{Zeta}[-m-1] + \text{HurwitzZeta}[-m-1, 2p] + \text{HurwitzZeta}[-m-1, 2q] + \left(1-2^{-m}\right)\text{HurwitzZeta}[-m-1, 2p+2q-1]\right) - 2p\,\text{HurwitzZeta}[-m, 2p] - 2q\,\text{HurwitzZeta}[-m, 2q] - 2\left(1-2^{-m}\right)(p+q)\,\text{HurwitzZeta}[-m, -1+2p+2q]\right)n^{-m} \text{ // FullSimplify}$$

Out[ ]= True

In[ ]:=
```
TMP05 = 5/4 - 3 p + 2 p^2 - 3 q + 4 p q + 2 q^2 + 2 (-1/12 + 1/2 HurwitzZeta[-1, -1 + 2 p + 2 q]) +
  13 Log[2]/12 - 2 n Log[2] + n^2 Log[2] - 2 p^2 Log[2] - 4 p q Log[2] - 2 q^2 Log[2] -
  3 Log[Glaisher] - Log[n]/4 - n Log[n] + 2 p^2 Log[n] + 2 q^2 Log[n] - 2 p Log[Gamma[2 p]] -
  2 q Log[Gamma[2 q]] + PolyGamma[-2, 2 p] + PolyGamma[-2, 2 q] +
  PoincareSum[(-1)^(m-1)/m (1/(m + 1) (Zeta[-m - 1] + HurwitzZeta[-m - 1, 2 p] + HurwitzZeta[-m - 1, 2 q] +
        (1 - 2^-m) HurwitzZeta[-m - 1, 2 p + 2 q - 1]) - 2 p HurwitzZeta[-m, 2 p] -
      2 q HurwitzZeta[-m, 2 q] - 2 (1 - 2^-m) (p + q) HurwitzZeta[-m, -1 + 2 p + 2 q]) n^-m, {m, 1, ∞}];
```

Leading terms

In[ ]:=
```
TMP =
 5/4 - 3 p + 2 p^2 - 3 q + 4 p q + 2 q^2 + 2 (-1/12 + 1/2 HurwitzZeta[-1, -1 + 2 p + 2 q]) + 13 Log[2]/12 - 2 n Log[2] +
   n^2 Log[2] - 2 p^2 Log[2] - 4 p q Log[2] - 2 q^2 Log[2] - 3 Log[Glaisher] - Log[n]/4 - n Log[n] +
   2 p^2 Log[n] + 2 q^2 Log[n] - 2 p Log[Gamma[2 p]] - 2 q Log[Gamma[2 q]] +
   PolyGamma[-2, 2 p] + PolyGamma[-2, 2 q] // FunctionExpand // Expand
```

Out[ ]=
```
13 Log[2]/12 - 2 n Log[2] + n^2 Log[2] - 2 p^2 Log[2] - 4 p q Log[2] -
 2 q^2 Log[2] - 3 Log[Glaisher] - Log[n]/4 - n Log[n] + 2 p^2 Log[n] + 2 q^2 Log[n] -
 2 p Log[Gamma[2 p]] - 2 q Log[Gamma[2 q]] + PolyGamma[-2, 2 p] + PolyGamma[-2, 2 q]
```

Auxiliary results

In[ ]:=
```
13 Log[2]/12 - 2 p^2 Log[2] - 4 p q Log[2] - 2 q^2 Log[2] - 3 Log[Glaisher] -
  Log[n]/4 + 2 p^2 Log[n] + 2 q^2 Log[n] - 2 p Log[Gamma[2 p]] - 2 q Log[Gamma[2 q]] +
  PolyGamma[-2, 2 p] + PolyGamma[-2, 2 q] // Collect[#, {Log[n], Log[2]}] &
```

Out[ ]=
```
(13/12 - 2 p^2 - 4 p q - 2 q^2) Log[2] - 3 Log[Glaisher] + (-1/4 + 2 p^2 + 2 q^2) Log[n] -
 2 p Log[Gamma[2 p]] - 2 q Log[Gamma[2 q]] + PolyGamma[-2, 2 p] + PolyGamma[-2, 2 q]
```

```
(13/12 - 2 p^2 - 4 p q - 2 q^2) Log[2] - 3 Log[Glaisher] - 2 p Log[Gamma[2 p]] -
  2 q Log[Gamma[2 q]] + PolyGamma[-2, 2 p] + PolyGamma[-2, 2 q]
```

```
In[•]:= (13/12 - 2 p^2 - 4 p q - 2 q^2) Log[2] == -2 ((p + q)^2 - 13/24) Log[2] // FullSimplify
Out[•]= True
```

```
In[•]:= TMP == Log[2] n^2 - n Log[n] - 2 Log[2] n + 2 (+ p^2 + q^2 - 1/8) Log[n] -
    2 ((p + q)^2 - 13/24) Log[2] - 3 Log[Glaisher] - 2 p Log[Gamma[2 p]] +
    PolyGamma[-2, 2 p] - 2 q Log[Gamma[2 q]] + PolyGamma[-2, 2 q] // FullSimplify
Out[•]= True
```

```
In[•]:= TMP06 = Log[2] n^2 - n Log[n] - 2 Log[2] n + 2 (+ p^2 + q^2 - 1/8) Log[n] -
    2 ((p + q)^2 - 13/24) Log[2] - 3 Log[Glaisher] - 2 p Log[Gamma[2 p]] +
    PolyGamma[-2, 2 p] - 2 q Log[Gamma[2 q]] + PolyGamma[-2, 2 q] +
    PoincareSum[(-1)^(m-1)/m (1/(m + 1) (Zeta[-m - 1] + HurwitzZeta[-m - 1, 2 p] + HurwitzZeta[-m - 1, 2 q] +
          (1 - 2^-m) HurwitzZeta[-m - 1, 2 p + 2 q - 1]) - 2 p HurwitzZeta[-m, 2 p] -
        2 q HurwitzZeta[-m, 2 q] - 2 (1 - 2^-m) (p + q) HurwitzZeta[-m, -1 + 2 p + 2 q]) n^-m, {m, 1, ∞}];
```

Formula

In[ ]:=

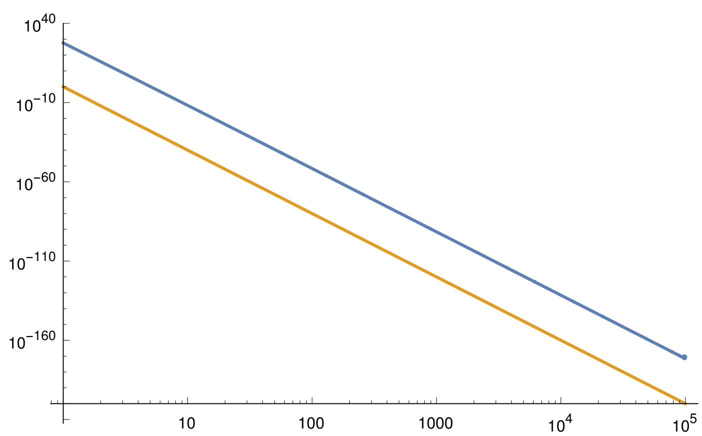
```
ASYMPE0[n_, q_, p_] :=
  Log[2] n^2 - n Log[n] - 2 Log[2] n + 2 (+ p^2 + q^2 - 1/8) Log[n] - 2 ((p + q)^2 - 13/24) Log[2] -
   3 Log[Glaisher] - 2 p Log[Gamma[2 p]] + PolyGamma[-2, 2 p] - 2 q Log[Gamma[2 q]] +
   PolyGamma[-2, 2 q] + PoincareSum[(-1)^(m-1)/m (1/(m + 1) (Zeta[-m - 1] + HurwitzZeta[-m - 1, 2 p] +
          HurwitzZeta[-m - 1, 2 q] + (1 - 2^-m) HurwitzZeta[-m - 1, 2 p + 2 q - 1]) -
        2 p HurwitzZeta[-m, 2 p] - 2 q HurwitzZeta[-m, 2 q] -
        2 (1 - 2^-m) (p + q) HurwitzZeta[-m, -1 + 2 p + 2 q]) n^-m, {m, 1, ∞}];
```

Cross Check

```
2 (n - 1) Logλ[n, α, β] - LogD[n, α, β]; (* for comparison *)
```

In[ ]:=
```
λ[n_, α_, β_] := 2^-n Binomial[2 n + α + β, n];
Discr[n_, α_, β_] := 2^(-n (n-1)) Product[v^(v-2 n+2) (v + α)^(v-1) (v + β)^(v-1) (v + n + α + β)^(n-v), {v, 1, n}];

n =.; (*  *)

p = 1 / 2;
q = π;

α = 2 p - 1;
β = 2 q - 1;


REFLogλ = - Log[2] n + Log[Gamma[2 n + α + β + 1]] - Log[Gamma[n + α + β + 1]] - Log[Gamma[n + 1]];
REFLogD = - n (n - 1) Log[2] + FracA2[n] + FracB2[n, α] + FracB2[n, β] + FracC2[n, α + β];

REF = 2 (n - 1) REFLogλ - REFLogD;

K = 40;

ASYMP = ASYMPE0[n, q, p] /. PoincareSumNormalize[K - 1];

LogLogPlot[{Abs[REF - ASYMP], n^-K}, {n, 1, 100 000}, WorkingPrecision → 512]

Clear[α, β, n, p, q, zeros, K];
```

Out[ ]=



Proof of Theorem 1.4

Cross check of starting point

Formula clear by definition of logarithmic energy, potential energy, and definition of discriminant.

```
In[ ]:= n = M - 2; (* M instead of N which is used in Mathematica  *)

p = 1;
q = 1;

α = 2 p - 1;
β = 2 q - 1;


REFLogλ = - Log[2] n + Log[Gamma[2 n + α + β + 1]] - Log[Gamma[n + α + β + 1]] - Log[Gamma[n + 1]];
REFLogD = - n (n - 1) Log[2] + FracA2[n] + FracB2[n, α] + FracB2[n, β] + FracC2[n, α + β];
REFLogP[n_, α_, β_] := - Log[Gamma[α + 1]] + Log[Gamma[n + α + 1]] - Log[Gamma[n + 1]];

REF = 2 (n + 1) REFLogλ - REFLogD - 4 REFLogP[n, α, β] - 2 Log[2];

ΔM = 2^(M (M-1)) M^M Product[k^(3 k), {k, 1, M - 1}] × Product[k^(-k), {k, M - 1, 2 (M - 1)}];

RES = -Log[ΔM];

Table[RES == REF // FullSimplify, {M, 2, 10}]

Clear[α, β, n, p, q, zeros, K];

Out[ ]= {True, True, True, True, True, True, True, True, True}
```

```
In[ ]:= n = M - 2; (* M instead of N which is used in Mathematica  *)

        p = 1;
        q = 1;

        α = 2 p - 1;
        β = 2 q - 1;


        REFLogλ = - Log[2] n + Log[Gamma[2 n + α + β + 1]] - Log[Gamma[n + α + β + 1]] - Log[Gamma[n + 1]];
        REFLogD = - n (n - 1) Log[2] + FracA2[n] + FracB2[n, α] + FracB2[n, β] + FracC2[n, α + β];
        REFLogP[n_, α_, β_] := - Log[Gamma[α + 1]] + Log[Gamma[n + α + 1]] - Log[Gamma[n + 1]];

        REF = 2 (n + 1) REFLogλ - REFLogD - 4 REFLogP[n, α, β] - 2 Log[2];


        RES = -M (M - 1) Log[2] - M Log[M] + 3 Zeta^(1,0)[-1, 1] -
            3 Zeta^(1,0)[-1, M] - Zeta^(1,0)[-1, M - 1] + Zeta^(1,0)[-1, 2 M - 1];

        Table[RES == REF // FullSimplify, {M, 2, 10}]

        Clear[α, β, n, p, q, zeros, K];
Out[ ]= {True, True, True, True, True, True, True, True, True}
```

Application of asymptotics

```
-M (M - 1) Log[2] - M Log[M] + 3 Zeta^(1,0)[-1, 1] -
   3 Zeta^(1,0)[-1, M] - Zeta^(1,0)[-1, M - 1] + Zeta^(1,0)[-1, 2 M - 1];

AsymptoticsLogGamma[a, x];
AsymptoticsHurwitzZetaPrime[a, x];

(* for comparison ... *)
```

In[ ]:= **TMP01 = -M (M - 1) Log[2] - M Log[M] + 3 Zeta$^{(1,0)}$[-1, 1] - 3 AsymptoticsHurwitzZetaPrime[0, M] - AsymptoticsHurwitzZetaPrime[-1, M] + AsymptoticsHurwitzZetaPrime[-1, 2 M] // Expand**

Clear[α, β];

Out[ ]= $-\frac{1}{4} + M\,\text{Log}[2] - M^2\,\text{Log}[2] - \frac{4\,\text{Log}[M]}{3} + 2\,M\,\text{Log}[M] - 2\,M^2\,\text{Log}[M] + \frac{13}{12}\,\text{Log}[2\,M] - 3\,M\,\text{Log}[2\,M] + 2\,M^2\,\text{Log}[2\,M] - \text{PoincareSum}\left[\frac{(-1)^m\,M^{-m}\,\text{HurwitzZeta}[-1-m,\,-1]}{m\,(1+m)},\,\{m,\,1,\,\infty\}\right] + \text{PoincareSum}\left[\frac{\left(-\frac{1}{2}\right)^m\,M^{-m}\,\text{HurwitzZeta}[-1-m,\,-1]}{m\,(1+m)},\,\{m,\,1,\,\infty\}\right] - 3\,\text{PoincareSum}\left[\frac{(-1)^m\,M^{-m}\,\text{HurwitzZeta}[-1-m,\,0]}{m\,(1+m)},\,\{m,\,1,\,\infty\}\right] + 3\,\text{Zeta}^{(1,0)}[-1,\,1]$

Simplification

In[ ]:= **TMP02 = TMP01 /. PoincareSumFactorUnderSum**

Out[ ]= $-\frac{1}{4} + M\,\text{Log}[2] - M^2\,\text{Log}[2] - \frac{4\,\text{Log}[M]}{3} + 2\,M\,\text{Log}[M] - 2\,M^2\,\text{Log}[M] + \frac{13}{12}\,\text{Log}[2\,M] - 3\,M\,\text{Log}[2\,M] + 2\,M^2\,\text{Log}[2\,M] + \text{PoincareSum}\left[\frac{(-1)^{1+m}\,M^{-m}\,\text{HurwitzZeta}[-1-m,\,-1]}{m\,(1+m)},\,\{m,\,1,\,\infty\}\right] + \text{PoincareSum}\left[\frac{\left(-\frac{1}{2}\right)^m\,M^{-m}\,\text{HurwitzZeta}[-1-m,\,-1]}{m\,(1+m)},\,\{m,\,1,\,\infty\}\right] + \text{PoincareSum}\left[-\frac{3\,(-1)^m\,M^{-m}\,\text{HurwitzZeta}[-1-m,\,0]}{m\,(1+m)},\,\{m,\,1,\,\infty\}\right] + 3\,\text{Zeta}^{(1,0)}[-1,\,1]$

In[ ]:= **TMP02 //. PoincareSumCollect**

Out[ ]= $-\frac{1}{4} + M\,\text{Log}[2] - M^2\,\text{Log}[2] - \frac{4\,\text{Log}[M]}{3} + 2\,M\,\text{Log}[M] - 2\,M^2\,\text{Log}[M] + \frac{13}{12}\,\text{Log}[2\,M] - 3\,M\,\text{Log}[2\,M] + 2\,M^2\,\text{Log}[2\,M] + \text{PoincareSum}\left[\frac{(-1)^{1+m}\,M^{-m}\,\text{HurwitzZeta}[-1-m,\,-1]}{m\,(1+m)} + \frac{\left(-\frac{1}{2}\right)^m\,M^{-m}\,\text{HurwitzZeta}[-1-m,\,-1]}{m\,(1+m)} - \frac{3\,(-1)^m\,M^{-m}\,\text{HurwitzZeta}[-1-m,\,0]}{m\,(1+m)},\,\{m,\,1,\,\infty\}\right] + 3\,\text{Zeta}^{(1,0)}[-1,\,1]$

```
TMP03 = -1/4 + M Log[2] - M^2 Log[2] - (4 Log[M])/3 +
   2 M Log[M] - 2 M^2 Log[M] + 13/12 Log[2 M] - 3 M Log[2 M] + 2 M^2 Log[2 M] +
   PoincareSum[((-1)^(1+m) M^-m HurwitzZeta[-1 - m, -1])/(m (1 + m)) + ((-1/2)^m M^-m HurwitzZeta[-1 - m, -1])/(m (1 + m)) -
     (3 (-1)^m M^-m HurwitzZeta[-1 - m, 0])/(m (1 + m)), {m, 1, ∞}] + 3 Zeta^(1,0)[-1, 1];

In[ ]:= TMP = ((-1)^(1+m) M^-m HurwitzZeta[-1 - m, -1])/(m (1 + m)) +
     ((-1/2)^m M^-m HurwitzZeta[-1 - m, -1])/(m (1 + m)) - (3 (-1)^m M^-m HurwitzZeta[-1 - m, 0])/(m (1 + m)) // Factor

Out[ ]= 1/(m (1 + m)) (-1)^(1+m) 2^-m M^-m
     (-HurwitzZeta[-1 - m, -1] + 2^m HurwitzZeta[-1 - m, -1] + 3 × 2^m HurwitzZeta[-1 - m, 0])

In[ ]:= 2^-m (-HurwitzZeta[-1 - m, -1] + 2^m HurwitzZeta[-1 - m, -1] + 3 × 2^m HurwitzZeta[-1 - m, 0]) //
     Expand

In[ ]:= (HurwitzZeta[-1 - m, -1] - 2^-m HurwitzZeta[-1 - m, -1] // Simplify) + 3 HurwitzZeta[-1 - m, 0]

In[ ]:= (2^-m (-1 + 2^m) // Distribute) HurwitzZeta[-1 - m, -1] + 3 HurwitzZeta[-1 - m, 0]

Out[ ]= (1 - 2^-m) HurwitzZeta[-1 - m, -1] + 3 HurwitzZeta[-1 - m, 0]

In[ ]:= TMP == (-1)^(m-1)/(m (1 + m)) (3 HurwitzZeta[-1 - m, 0] + (1 - 2^-m) HurwitzZeta[-1 - m, -1]) M^-m // FullSimplify

Out[ ]= True
```

HurwitzZeta functions of given type can be changed to Bernoulli numbers and polynomials ...

```
In[ ]:= 3 HurwitzZeta[-1 - m, 0] + (1 - 2^-m) HurwitzZeta[-1 - m, -1] // FullSimplify

Out[ ]= 3 (-1)^m + (4 - 2^-m) HurwitzZeta[-1 - m, -1]

In[ ]:= Table[3 HurwitzZeta[-1 - m, 0] + (1 - 2^-m) HurwitzZeta[-1 - m, -1] ==
     3 (-1)^m + (4 - 2^-m) HurwitzZeta[-1 - m, -1], {m, 1, 10}]

Out[ ]= {True, True, True, True, True, True, True, True, True, True}
```

In[ ]:=
```
Table[3 HurwitzZeta[-1 - m, 0] + (1 - 2^-m) HurwitzZeta[-1 - m, -1] ==
  -3 BernoulliB[m + 2]/(m + 2) - (1 - 2^-m) BernoulliB[m + 2, -1]/(m + 2), {m, 1, 10}]
```
Out[ ]= {True, True, True, True, True, True, True, True, True, True}

In[ ]:=
```
3 HurwitzZeta[-1 - m, 0] + (1 - 2^-m) HurwitzZeta[-1 - m, -1] ==
  -3 BernoulliB[m + 2]/(m + 2) - (1 - 2^-m) BernoulliB[m + 2, -1]/(m + 2) //
 FullSimplify[#, Assumptions -> {m ∈ Integers, m ≥ 0}] &
```
Out[ ]= True

Since

In[ ]:=
```
BernoulliB[n, -1] == BernoulliB[n] + (-1)^n n // FullSimplify[#, Assumptions -> {n ∈ Integers}] &
```
Out[ ]= True

In[ ]:=
```
-3 BernoulliB[m + 2]/(m + 2) - (1 - 2^-m) BernoulliB[m + 2, -1]/(m + 2) ==
  (-1)^(m-1) ((1 - 2^-m) + (4 - 2^-m) BernoulliB[m + 2]/(m + 2)) //
 FullSimplify[#, Assumptions -> {m ∈ Integers, m ≥ 0}] &
```
Out[ ]= $(-1+(-1)^m)$ BernoulliB[2 + m] == 0

... which is true for integers >= 0.

In[ ]:=
```
TMP == 1/(m (1 + m)) ((1 - 2^-m) + (4 - 2^-m) BernoulliB[m + 2]/(m + 2)) M^-m //
 FullSimplify[#, Assumptions -> {m ∈ Integers, m ≥ 0}] &
```
Out[ ]= $\frac{(-1+(-1)^m)\, M^{-m}\, \text{BernoulliB}[2+m]}{m}$ == 0

```
TMP04 = -1/4 + M Log[2] - M^2 Log[2] - 4 Log[M]/3 + 2 M Log[M] -
  2 M^2 Log[M] + 13/12 Log[2 M] - 3 M Log[2 M] + 2 M^2 Log[2 M] + 3 Zeta^(1,0)[-1, 1] +
  PoincareSum[1/(m (1 + m)) ((1 - 2^-m) + (4 - 2^-m) BernoulliB[m + 2]/(m + 2)) M^-m, {m, 1, ∞}];
```

Leading terms

```
In[ ]:= TMP = -1/4 + M Log[2] - M^2 Log[2] - (4 Log[M])/3 + 2 M Log[M] - 2 M^2 Log[M] +
        13/12 Log[2 M] - 3 M Log[2 M] + 2 M^2 Log[2 M] + 3 Zeta^(1,0)[-1, 1] // FullSimplify

Out[ ]= (13 Log[2])/12 + M^2 Log[2] - 3 Log[Glaisher] - Log[M]/4 - M Log[4 M]

In[ ]:= TMP == Log[2] M^2 - M Log[ M] - 2 Log[2] M - 1/4 Log[M] + (13 Log[2])/12 - 3 Log[Glaisher] // FullSimplify

Out[ ]= True

In[ ]:= TMP05 = Log[2] M^2 - M Log[ M] - 2 Log[2] M - 1/4 Log[M] + (13 Log[2])/12 - 3 Log[Glaisher] +
        PoincareSum[1/(m (1 + m)) ((1 - 2^-m) + (4 - 2^-m) BernoulliB[m + 2]/(m + 2)) M^-m, {m, 1, ∞}];
```

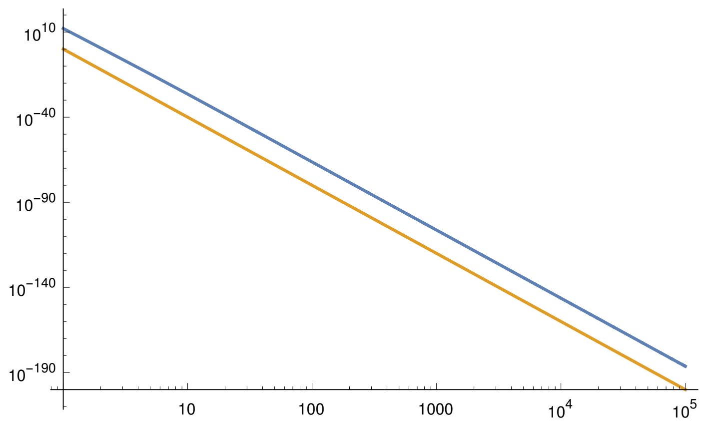
Formula

```
In[ ]:= ASYMPminE0[M_] := Log[2] M^2 - M Log[ M] - 2 Log[2] M - 1/4 Log[M] + (13 Log[2])/12 - 3 Log[Glaisher] +
        PoincareSum[1/(m (1 + m)) ((1 - 2^-m) + (4 - 2^-m) BernoulliB[m + 2]/(m + 2)) M^-m, {m, 1, ∞}];
```

Cross Check

In[ ]:=
```
REF = -M (M - 1) Log[2] - M Log[M] + 3 Zeta^(1,0)[-1, 1] -
    3 Zeta^(1,0)[-1, M] - Zeta^(1,0)[-1, M - 1] + Zeta^(1,0)[-1, 2 M - 1];

K = 40;

ASYMP = ASYMPminE0[M] /. PoincareSumNormalize[K - 1];

LogLogPlot[{Abs[REF - ASYMP], M^-K}, {M, 1, 100 000}, WorkingPrecision → 512]

Clear[α, β, n, p, q, zeros, K];
```

Out[ ]=



Misc

Katsurada' s Formula

In[ ]:=
```
modStirlingS[j_, k_, x_] := 1/j! D[(1 - z)^-s (-Log[1 - z])^j, {z, k}] /. z → 0;
```

In[ ]:=
```
AuxP[m_, s_, w_] := Sum[((s - 1) w)^j / j!, {j, 0, m}];
AuxQ[m_, k_, s_, w_] := Sum[modStirlingS[m - j, k, s] / j! (-w)^j, {j, 0, m}];
```