Weakly Non-Linear Shape Oscillations of Inviscid Drops - Supplement

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The present supplementary material to our above paper details the determination of the velocity and pressure fields, as well as surface deformation amplitudes, of oscillating drops. The supplement is designed as a collection of equations for coefficients and deformation amplitudes in the solutions for the various orders of approximation. We have also put together equations arising in the application of the orthogonality of Legendre polynomials for the determination of coefficients. Having removed these details from the bulk of the paper text should improve its readability. The structures of the various solutions are kept in the paper.

Appendix A

A.1. *Orthogonality of the Legendre polynomials*

Integrals of products of Legendre functions $P_l^n(x)$ appear in the coefficient calculation from the boundary conditions. The degree l may assume all natural numbers, and the order n is either zero or unity. The integrals are evaluated accounting for the orthogonality of the Legendre functions over the interval $[-1,1]$ of the independent variable x. Integrals of products of two Legendre functions are

$$\int_{-1}^{1} P_{l_1}^n(x) P_{l_2}^n(x) dx = \frac{2\,(l_1+n)!}{(2l_1+1)\,(l_1-n)!}\delta_{l_1 l_2}\ , \tag{A.1}$$

where $\delta_{l_1 l_2}$ is the Kronecker delta.

Multiplication of the boundary conditions with a Legendre function and integration produces integrals of products of three or four Legendre functions of different degrees. For the special case of three Legendre polynomials ($n=0$), integrals of the form

$$\int_{-1}^{1} P_{l_1}(x) P_{l_2}(x) P_{l_3}(x) dx = 2\begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}^2 \tag{A.2}$$

occur. The 3-j symbols on the right of this equation can be computed using the Racah formula

$$\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} = (-1)^{a-b-\gamma}\sqrt{\Delta(a,b,c)}\sqrt{(a+\alpha)!(a-\alpha)!(b+\beta)!(b-\beta)!(c+\gamma)!(c-\gamma)!}$$
$$\times \sum_t \frac{(-1)^t}{x} \tag{A.3}$$

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where $\Delta(a,b,c)$ is a triangle coefficient defined by

$$\Delta(a,b,c) = \frac{(a+b-c)!(a-b+c)!(-a+b+c)!}{(a+b+c+1)!}\,, \tag{A.4}$$

(Shore & Menzel (1968), p. 273), and

$$x = t!(c-b+t+\alpha)!(c-a+t-\beta)!(a+b-c-t)!(a-t-\alpha)!(b-t+\beta)! \tag{A.5}$$

with an integer t, and the sum in (A.3) is over all integers t for which the factorials in x have non-negative arguments (Messiah (1962), p. 1058; Shore & Menzel (1968), p. 273). In particular, the number of terms is equal to $\nu+1$, where ν is the smallest of the nine numbers

$$a\pm\alpha, \quad b\pm\beta, \quad c\pm\gamma,$$
$$a+b-c, \quad b+c-a, \quad c+a-b$$

(Messiah (1962), p. 1058).

Upon multiplication with a Legendre function and integration, the right-hand sides of the boundary conditions may exhibit overlap integrals of three or four Legendre functions also. The general solutions by Dong & Lemus (2002) include all possible combinations of degrees and orders which occur in this analysis. The overlap integral for three Legendre functions is

$$\int_{-1}^{1} P_{l_1}^{m_1}(x)P_{l_2}^{m_2}(x)P_{l_3}^{m_3}(x)dx = \sqrt{\frac{(l_1+m_1)!\,(l_2+m_2)!\,(l_3+m_3)!}{(l_1-m_1)!\,(l_2-m_2)!\,(l_3-m_3)!}} \tag{A.6}$$

$$\times \sum_{l_{12}}\sum_{l_{123}} G_{12}G_{123} \times \sqrt{\frac{(l_{123}-m_{123})!}{(l_{123}+m_{123})!}} I\left(l_{123}, m_{123}\right)$$

while for four Legendre functions it reads

$$\int_{-1}^{1} P_{l_1}^{m_1}(x)P_{l_2}^{m_2}(x)P_{l_3}^{m_3}(x)P_{l_4}^{m_4}(x)dx = \sqrt{\frac{(l_1+m_1)!\,(l_2+m_2)!\,(l_3+m_3)!\,(l_4+m_4)!}{(l_1-m_1)!\,(l_2-m_2)!\,(l_3-m_3)!\,(l_4-m_4)!}}$$

$$\times \sum_{l_{12}}\sum_{l_{123}}\sum_{l_{1234}} G_{12}G_{123}G_{1234}\sqrt{\frac{(l_{1234}-m_{1234})!}{(l_{1234}+m_{1234})!}} I\left(l_{1234}, m_{1234}\right) \tag{A.7}$$

The unknown coefficients G are defined as

$$G_{12} = (-1)^{m_{12}}\left(2l_{12}+1\right)\begin{pmatrix} l_1 & l_2 & l_{12} \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} l_1 & l_2 & l_{12} \\ m_1 & m_2 & -m_{12} \end{pmatrix} \tag{A.8}$$

where the summation indexes $|l_1-l_2| \leqslant l_{12} \leqslant l_1+l_2$ and $l_{12} \geqslant m_{12}$, defining $m_{12} = \sum_{i=1}^{2} m_i$,

$$G_{123} = (-1)^{m_{123}}\left(2l_{123}+1\right)\begin{pmatrix} l_{12} & l_3 & l_{123} \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} l_{12} & l_3 & l_{123} \\ m_{12} & m_3 & -m_{123} \end{pmatrix} \tag{A.9}$$

where $|l_{12}-l_3| \leqslant l_{123} \leqslant l_{12}+l_3$ and $l_{123} \geqslant m_{123}$, defining $m_{123} = \sum_{i=1}^{3} m_i$, and

$$G_{1234} = (-1)^{m_{1234}}\left(2l_{1234}+1\right)\begin{pmatrix} l_{123} & l_4 & l_{1234} \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} l_{123} & l_4 & l_{1234} \\ m_{123} & m_4 & -m_{1234} \end{pmatrix} \tag{A.10}$$

where $|l_{123}-l_4| \leqslant l_{1234} \leqslant l_{123}+l_4$ and $l_{1234} \geqslant m_{1234}$, defining $m_{1234} = \sum_{i=1}^{4} m_i$. The coefficient $I(l_{123}, m_{123})$ is defined as

$$I(l_{123}, m_{123}) = \frac{\left((-1)^{m_{123}} + (-1)^{l_{123}}\right) 2^{m_{123}-2} m_{123} \Gamma\left((l_{123})/2\right) \Gamma\left((l_{123} + m_{123} + 1)/2\right)}{((l_{123} - m_{123})/2)! \Gamma\left((l_{123} + 3)/2\right)} \tag{A.11}$$

where Γ is the gamma function. The same definition applies for $I(l_{1234}, m_{1234})$, switching the l_{123} to l_{1234} and m_{123} to m_{1234}. The integrals of the Legendre polynomials are calculated using the 3-j symbols. They appear in the coefficients $G_{12...i}$ and in relation $(A.2)$. They are zero whenever the triangle inequality between the integers of the first row is not satisfied. This property terminates the expansion of the coefficients in all boundary conditions. Following the same example, the zero value occurs when $m_1 + m_2 + m_3 \neq 0$, or when $a = b = c = 0$ and simultaneously $l_1 + l_2 + l_3 = 2\lambda + 1$ appears, where $\lambda = 0, 1, 2, ...$ This property eliminates the odd-order Legendre polynomials from the second-order approximation.

A.2. *Second-order solutions*

The coefficients C_{21l} in the second-order solutions "21" are calculated using the second-order zero normal stress boundary condition (2.25). We substitute the pressure solution p_{21} and the known η_{21} from (3.22) and (3.28), respectively, into the second-order zero normal stress boundary condition. The first-order solutions are also substituted into (2.25), setting r to 1. After applying the orthogonality of the Legendre polynomials, the coefficients C_{21l} for every l between zero and L, and for the time dependencies according to $\exp\left[-2\alpha_m^{(p)}\tau\right]$ and $\exp\left[-2\alpha_m^{(n)}\tau\right]$, read

$$C_{21l} = \frac{2l+1}{2}\left[\left(\frac{m(m+1)-1}{2} - \frac{3}{8}\alpha_{m,0}^2\right)\int_{-1}^{1} P_m(x)^2 P_l(x)dx - \frac{\alpha_{m,0}^2}{8m^2}\int_{-1}^{1} P_m^1(x)^2 P_l(x)dx\right] - H_{21l}(l-1)(l+2) \tag{A.12}$$

The integrals appearing in this equation are solved with the method presented in Dong & Lemus (2002), as detailed in section A.1 above. From the equations for the second-order coefficients follows that $C_{21l}^{(p)} = C_{21l}^{(n)}$. The coefficient $C_{21l}^{(pn)}$ was set to zero for every l in an earlier step of the derivation of the solutions already.

For the contributions "22" to the second-order solutions, the coefficients C_{22k} are deduced from the homogeneous kinematic boundary condition for each time dependency and each summation index k

$$C_{22k}^{(p)} = \frac{\hat{\eta}_{22k}^{(p)}\alpha_{2k}^{(p)}}{k(k+1)} \; ; \quad C_{22k}^{(n)} = \frac{\hat{\eta}_{22k}^{(n)}\alpha_{2k}^{(n)}}{k(k+1)} \tag{A.13}$$

The two amplitudes $\hat{\eta}_{22k}^{(p)}$ and $\hat{\eta}_{22k}^{(n)}$ are determined from the second-order initial conditions. For values $k \geqslant 1$ they read (with H_{21k} taken from (3.28))

$$\begin{aligned}\hat{\eta}_{22k}^{(p)} &= \frac{\alpha_{2k}^{(n)}\left(H_{21k}^{(p)} + H_{21k}^{(n)} + H_{21k}^{(pn)}\right) - \left(2\alpha_m^{(p)}H_{21k}^{(p)} + 2\alpha_m^{(n)}H_{21k}^{(n)} + (\alpha_m^{(p)} + \alpha_m^{(n)})H_{21k}^{(pn)}\right)}{\alpha_{2k}^{(p)} - \alpha_{2k}^{(n)}} \\ &= \frac{\alpha_{2k}^{(n)}\left(H_{21k}^{(p)} + H_{21k}^{(n)} + H_{21k}^{(pn)}\right) - \left(2\alpha_m^{(p)}H_{21k}^{(p)} + 2\alpha_m^{(n)}H_{21k}^{(n)}\right)}{\alpha_{2k}^{(p)} - \alpha_{2k}^{(n)}}\end{aligned} \tag{A.14}$$

$$\hat{\eta}^{(n)}_{22k} = -\frac{\alpha^{(p)}_{2k}\left(H^{(p)}_{21k} + H^{(n)}_{21k} + H^{(pn)}_{21k}\right) - \left(2\alpha^{(p)}_m H^{(p)}_{21k} + 2\alpha^{(n)}_m H^{(n)}_{21k} + (\alpha^{(p)}_m + \alpha^{(n)}_m) H^{(pn)}_{21k}\right)}{\alpha^{(p)}_{2k} - \alpha^{(n)}_{2k}}$$

$$= -\frac{\alpha^{(p)}_{2k}\left(H^{(p)}_{21k} + H^{(n)}_{21k} + H^{(pn)}_{21k}\right) - \left(2\alpha^{(p)}_m H^{(p)}_{21k} + 2\alpha^{(n)}_m H^{(n)}_{21k}\right)}{\alpha^{(p)}_{2k} - \alpha^{(n)}_{2k}} \tag{A.15}$$

while for $k = 0$ we have

$$\hat{\eta}^{(p)}_{220} = \hat{\eta}^{(n)}_{220} = -\frac{1}{2}\left(H^{(p)}_{210} + H^{(n)}_{210} + H^{(pn)}_{210} + \frac{1}{2m+1}\right) \tag{A.16}$$

Knowing that $\hat{\eta}^{(p)}_{22k} = \hat{\eta}^{(n)}_{22k} =: \hat{\eta}_{22k}$, and that the frequencies $\alpha^{(p)}_{2k}$ and $\alpha^{(n)}_{2k}$ are complex conjugate, allows the contributions "22" to the second-order solutions to be formulated in a simple form.

A.3. *Coefficients for third-order solutions "31"*

In the simplified forms of the third-order solutions, the relations between the coefficients are

$$C^{(p)}_{31h} = C^{(n)}_{31h} =: C^{(1)}_{31h} \quad C^{(ppn)}_{31h} = C^{(pnn)}_{31h} =: C^{(2)}_{31h} \tag{A.17}$$

$$C^{(pp)}_{31hk} = C^{(nn)}_{31hk} =: C^{(3)}_{31hk} \quad C^{(pn)}_{31hk} = C^{(np)}_{31hk} =: C^{(4)}_{31hk} \tag{A.18}$$

$$H^{(p)}_{31h} = H^{(n)}_{31h} =: H^{(1)}_{31h} \quad H^{(ppn)}_{31h} = H^{(pnn)}_{31h} =: H^{(2)}_{31h} \tag{A.19}$$

$$H^{(pp)}_{31hk} = H^{(nn)}_{31hk} =: H^{(3)}_{31hk} \quad H^{(pn)}_{31hk} = H^{(np)}_{31hk} =: H^{(4)}_{31hk}\,. \tag{A.20}$$

A.4. *Coefficients for the third-order solutions "32"*

The coefficients C_{32h} in the contributions "32" to the third-order solutions are deduced from the homogeneous kinematic boundary condition for each time dependency and every summation index k in the general solution as

$$C^{(p)}_{32h} = \frac{\hat{\eta}^{(p)}_{32h}\alpha^{(p)}_{3h}}{h(h+1)} \qquad C^{(n)}_{32h} = \frac{\hat{\eta}^{(n)}_{32h}\alpha^{(n)}_{3h}}{h(h+1)} \tag{A.21}$$

The coefficients $\hat{\eta}^{(p)}_{32h}$ and $\hat{\eta}^{(n)}_{32h}$ in the solutions remain to be determined. Using the third-order initial conditions, we can deduce the deformation amplitude of the drop surface for each time dependency and summation index h. The initial conditions relate the amplitudes of contribution "31" to "32". The amplitudes read

$$\begin{aligned}
\hat{\eta}^{(p)}_{32h} = \frac{1}{\alpha^{(p)}_{3h} - \alpha^{(n)}_{3h}} \Bigg[& \alpha^{(n)}_{3h}\Bigg(H^{(p)}_{31h} + H^{(n)}_{31h} \\
& + \sum_{k=0}^{K}\left(H^{(pp)}_{31hk} + H^{(nn)}_{31hk} + H^{(pn)}_{31hk} + H^{(np)}_{31hk}\right) + H^{(ppn)}_{31h} + H^{(pnn)}_{31h}\Bigg) \\
& - \Bigg(3\alpha^{(p)}_m H^{(p)}_{31h} + 3\alpha^{(n)}_m H^{(n)}_{31h} + (2\alpha^{(p)}_m + \alpha^{(n)}_m) H^{(pn)}_{31h} + (\alpha^{(p)}_m + 2\alpha^{(n)}_m) H^{(pnn)}_{31h} \\
& + \sum_{k=0}^{K}(\alpha^{(p)}_m + \alpha^{(p)}_{2k}) H^{(pp)}_{31hk} + \sum_{k=0}^{K}(\alpha^{(n)}_m + \alpha^{(n)}_{2k}) H^{(nn)}_{31hk} \\
& + \sum_{k=0}^{K}(\alpha^{(p)}_m + \alpha^{(n)}_{2k}) H^{(pn)}_{31hk} + \sum_{k=0}^{K}(\alpha^{(n)}_m + \alpha^{(p)}_{2k}) H^{(np)}_{31hk}\Bigg)\Bigg]
\end{aligned} \tag{A.22}$$

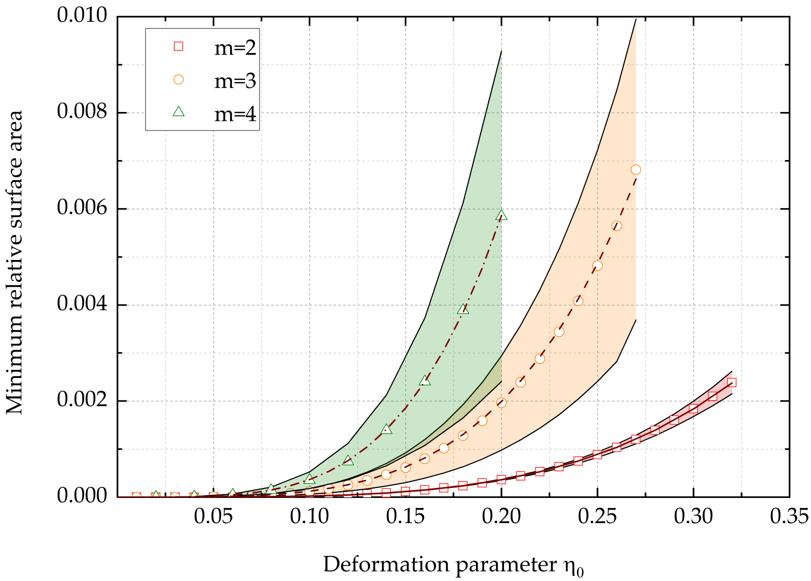

FIGURE 1. Minimum deviation of the drop surface from the spherical shape as a function of the deformation parameter η_0 for the modes of initial deformation $m = 2,\ 3$ and 4.

$$\hat{\eta}^{(n)}_{32h} = \frac{1}{\alpha^{(n)}_{3h} - \alpha^{(p)}_{3h}} \Bigg[\alpha^{(p)}_{3h} \Bigg(H^{(p)}_{31h} + H^{(n)}_{31h} + \sum_{k=0}^{K} \left(H^{(pp)}_{31hk} + H^{(nn)}_{31hk} + H^{(pn)}_{31hk} + H^{(np)}_{31hk} \right) + H^{(ppn)}_{31h} + H^{(pnn)}_{31h} \Bigg) + \Bigg(3\alpha^{(p)}_{m} H^{(p)}_{31h} + 3\alpha^{(n)}_{m} H^{(n)}_{31h} + (2\alpha^{(p)}_{m} + \alpha^{(n)}_{m}) H^{(pn)}_{31h} + (\alpha^{(p)}_{m} + 2\alpha^{(n)}_{m}) H^{(pnn)}_{31h} + \sum_{k=0}^{K} (\alpha^{(p)}_{m} + \alpha^{(p)}_{2k}) H^{(pp)}_{31hk} + \sum_{k=0}^{K} (\alpha^{(n)}_{m} + \alpha^{(n)}_{2k}) H^{(nn)}_{31hk} + \sum_{k=0}^{K} (\alpha^{(p)}_{m} + \alpha^{(n)}_{2k}) H^{(pn)}_{31hk} + \sum_{k=0}^{K} (\alpha^{(n)}_{m} + \alpha^{(p)}_{2k}) H^{(np)}_{31hk} \Bigg) \Bigg] \tag{A.23}$$

These equations show that $\hat{\eta}^{(p)}_{32h} = \hat{\eta}^{(n)}_{32h}$, which we denote $\hat{\eta}_{32h}$. Since the frequencies $\alpha^{(p)}_{3h}$ and $\alpha^{(n)}_{3h}$ are complex conjugate, furthermore, this allows for a simple form of the formulation of the contributions "32" to the third-order solutions, where the simplified notation for the amplitude is used.

A.5. *Minimum relative drop surface area*

The data in Fig. 1 show the minimum deviation of the drop surface area from the spherical state as a function of the deformation parameter η_0 for the modes of initial deformation $m = 2,\ 3$ and 4. The lines are fit curves, showing the proportionality of the surface area deviations to η_0^4.

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